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Research Paper

A New Taylor Series based Numerical Method: Simple, Reliable, and Promising

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Abstract. Taylor series method is a simple analytical method, which is accessible to all non-mathematician, has slow convergence. This paper develops a new Taylor series based numerical method to overcome the shortcoming of the Taylor series while maintaining its simplicity. Some examples are given, showing its reliability and efficiency. The proposed method is also proved to be extremely effective for initial value problems and boundary value problems. The method provides a universal approach to various highly non-linear problems, and it sheds a bright light on numerical theories for practical engineering applications.

Keywords: Taylor Series, Singular Boundary Value Problem, KDV equation, Burgers' Equation, System of Burgers Equation.

1. Introduction

For many years, the researchers used various real-life examples as test problems like Lane-Emden type equations, Burger equations, KDV equation, etc. to verify the numerical methods. In this work, we shall consider the following class of nonlinear ODEs and PDEs:

$$y''(x)+\frac{\alpha}{x}y'(x)=f\big(x,y(x)\big),\qquad 0<x<1,\qquad y(0)=c, y'(0)=d, \tag{1}$$

$$y''(x)+\frac{\alpha}{x}y'(x)=f\big(x,y(x)\big),\qquad 0<x<1,\qquad y'(0)=0, \beta_1 y(1)+\beta_2 y'(1)=d, \tag{2}$$

$$y_i''(x)+\frac{\alpha}{x}y_i'(x)+h_i(x,y_1(x),y_2(x))=0,\qquad 0<x<1,\qquad y_i(0)=c_i, y_i'(0)=d_i, i=1,2 \tag{3}$$

$$y_i''(x)+\frac{\alpha}{x}y_i'(x)+h_i(x,y_1(x),y_2(x))=0,\qquad 0<x<1,\qquad {y_i}'(0)=0, y_i(1)=c_i, i=1,2 \tag{4}$$

$$u_t+uu_x=\mu u_{xx}, 0\le x\le 1, t>0, u(x,0)=2x, \tag{5}$$

$$u_t-6uu_x+u_{xxx}=0, 0<x<1,\qquad t>0, u(x,0)=-\frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx}{2}\right), \tag{6}$$

$$\begin{cases} u_t+uu_x+vu_y=\dfrac{1}{Re}(u_{xx}+u_{yy}), u(x,y,0)=x+y, 0\le x\le 1, t>0,\\ v_t+uv_x+vv_y=\dfrac{1}{Re}(v_{xx}+v_{yy}), v(x,y,0)=x-y, 0\le x\le 1, t>0.\end{cases} \tag{7}$$

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Lane-Emden type equations arise in various physical phenomena that occur in astrophysics and mathematical physics like stellar structure, thermionic currents, thermal explosions, radiative cooling, CTC, etc. In this work, we focus on such models by considering the following equation:

$$y''(x)+\frac{\alpha}{x}y'(x)=f(x,y(x)), \qquad 0<x<1, \tag{8}$$

where, $\alpha\geq 0,\ f:\mathbb{R}\ \times\mathbb{R}\ \to\mathbb{R}$ subject to both initial and boundary conditions. The status of the theoretical and numerical work on Lane-Emden type equations are well known. Many authors such as Dunninger et al. [1, 2], Zhange et al. [3, 4], Pandey et al. [5-7], etc. used equation (8) to develop theoretical results and Russell et al. [8, 9], Chawla et al. [10-12], Jain et al. [13], Pandey [14, 15] applied finite difference technique to find the numerical solution. Apart from these techniques many authors applied different types of numerical methods like rational Legendre approximation technique [16], methods based on splines polynomials [17-20], and different types of collocation approaches [21-24], methods based on Legendre function [25, 26], Haar wavelets and other orthonormal polynomial wavelets [27-29], NSFD method [30], Optimal homotopy analysis method [31] and etc.

We also focus on Burgers' equation which is highly nonlinear and one-dimensional analogue of Navier Stokes equation. It has a long history [32] and huge number of articles are available on Burgers's equations, its various generalisations to various forms in one dimension, two dimension and as system of nonlinear PDEs. Since the exact solution of Burgers's equation fails for small viscosity, it has posed great challenges to researchers to find its analytical solution. Fay [33] gave its solution in a particular set up. Hopf [34], and Cole [35] computed the exact solutions by transforming the Burger's equation to heat equation. Group theoretic methods for calculating the solution of Burgers' equation with appropriate boundary and initial conditions is proposed by Abd-el-Malek [36]. We list some existing methods which has been used to compute the analytical solutions of Burgers' equation: Hopf and Cole transformation [34, 35], Group theoretic method [37], Adomian decomposition method [38], Variation iteration method [39, 40], Tanh-function method [41, 42], Taylor series solution [43, 44].

He [45] derived an analytical solution of a system of Lane-Emden equations by using the Taylor series method and computed closed form solution of a system of Lane-Emden equations subject to given initial conditions. After that, he applied this method on fractal Bratu-type equation [46] arising in the electro spinning process and third order boundary value problem [47] arising in Architectural Engineering to derive the approximate solution which gives better accuracy than other methods. A lot of investigations are still pending and to address some of these we consider singular BVP, KDV equation, and Burgers's equations. Our main aim is to extend the numerical results of He [45] and explore it further. We present several Mathematica codes for this method corresponding to IVP, BVP, and SBVP, coupled IVP, and coupled SBVP to find the approximate and exact solutions with the best accuracy. We test each Mathematica code by considering different real-life problems of the form defined in Eq. (8) and compare our results with existing numerical results. We also extend the numerical results to Burger's equations, KDV equation, and system of nonlinear PDEs corresponding to initial conditions.

We have summarised the paper in a total of six sections. In section 2, we describe the Taylor series method. We have listed Mathematica codes in section 3. Several test examples are presented in section 4. We have derived the exact solutions of Burgers' equations, KDV equation, and system of nonlinear two-dimensional Burgers' equation in section 5. Finally, we draw our main conclusion along with future scope in section 6.

2. Description of the Method

Let us assume that the solution $y(x)$ of equation (8) is n times differentiable at $x=0$ and can be written as in the following Taylor series expansion:

$$y^{Taylor}(x)\approx\sum_{i=0}^{n}\frac{y^i(0)}{i!}x^i, \tag{9}$$

where $y^i(0)$ are unknown coefficients which are to be determined. Now, differentiating equation (8) n times with respect to x, we have:

$$\tfrac{d^i}{dx^i}(\ x^\alpha y'(x))'+x^\alpha f(x,y(x))=0,\ \text{for}\ i=0,1,2,\dots,n, \tag{10}$$

and setting $x=0$. Therefore, by using initial conditions and equations (10), we can easily determine the unknown constants $y^i(0)$ for all $i=0,1,\dots,n$. Finally, the exact solution of equation (8) can be written as:

$$y(x)=\lim_{n\to\infty}y^{Taylor}(x)\ . \tag{11}$$

For better understanding, we consider the simple linear IVP:

$$y'(x)=\tfrac{1}{1+x}\ y(0)=0. \tag{12}$$

Differentiating equation (12), with respect to x, we have:

$$y''(x)=-\frac{1}{(1+x)^2},y'''(x)=\frac{2}{(1+x)^3},y^{iv}(x)=-\frac{6}{(1+x)^4}. \tag{13}$$

Therefore, by setting $x=0$, we have:

$$y'(0)=0,y''(0)=-1,y'''(0)=2,y^{iv}(0)=-6. \tag{14}$$

So, from equation (9), we have the first order, second order, third order and fourth order approximation are as follows:

$$y^{Taylor}(x)\approx x,x-\frac{x^2}{2},\qquad x-\frac{x^2}{2}+\frac{x^3}{3},x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ . \tag{15}$$

By similar analysis, for $n\to\infty$ we have the exact solution, which is:

$$y\ (x)=\ \ x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=\log(1+x)\,. \tag{16}$$

We provide an algorithm of the Taylor series method which is used to develop code for different real-life problems to compute the solution.

2.1. Algorithm

Step 1. Fix the value of n (number of terms of the Taylor series).
Step 2. Input the differential equation (8) and corresponding initial conditions.
Step 3. Differentiate equation (8) with respect to x up to n times and put $x = 0$.
Step 4. Identify the unknown constants $y^i(0)$ for all $i = 0,1,\dots,n$ and solve the system of equations as in Step 3.
Step 5. Substitute all the values of $y^i(0)$ for all $i = 0,1,\dots,n$ in equation (9) to get the solution.

3. Mathematica Codes

By using the algorithm 2.1 and Mathematica 11.3 software, we develop codes for this method corresponding to second-order IVP, BVP, coupled IVP, and coupled BVP to find the approximate and exact solutions with the best accuracy.

3.1. IVP

We consider the following initial value problem:

$$y''(x) + \tfrac{\alpha}{x} y'(x) + g\big(x, y(x)\big), \;\; 0 < x < 1, \qquad y(0) = c, y'(0) = d, \tag{17}$$

where c and d are constants and $g(x,y)$ be arbitrary function of x and y. Below we present a Mathematica code for Eq. (17) which gives Taylor series solution up to n^{th} terms:

```
n = 3; (*Order of the  Taylor series solution*)
α = 2;
f = x (y''[x]) + α y'[x] + x (g (x, y[x])); (*Differential equation of the form (1.1)*)
y[0] = c; (*Where c is constant/Initial condition*)
y'[0] = d; (*Where d is constant/Initial condition*)
For[i = 0, i ≤ n, i++, Print[F_i = D[f, {x, i}]]]
A = Table[F_i == 0, {i, n}] /. x → 0
A // MatrixForm
Derivativevalue = Table[D[y[x], {x, i + 1}], {i, n}] /. x → 0
Derivativevalueatorigin = Flatten[Solve[A, Derivativevalue]]
Derivativevalueatorigin // MatrixForm
Derivativefinaloutput = Table[Derivativevalue[[i]] /. Derivativevalueatorigin[[i]], {i, 1, n}]
Sum[(D[y[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Final Taylor series solution of equation (1.1)*)
```

3.2. BVP

We consider equation (8) subject to the boundary condition in the following form:

$$y''(x) + \frac{\alpha}{x} y'(x) + g\big(x, y(x)\big) = 0, \qquad 0 < x < 1, \quad y'(0) = 0, \beta_2 y'(1) + \beta_1 y(1) = d, \tag{18}$$

where d are constants and $g(x,y)$ be arbitrary function of x and y. Since the value of $y(0)$ is not known therefore we take $y(0) = c$. Again, we provide a Mathematica code for (18) which gives Taylor series solution up to n^{th} terms as a function of c and x. The value of c can be determined by using the boundary condition $\beta_2 y'(1) + \beta_1 y(1) = d$.

```
n = 2; (*Order of the Taylor series solution*)
α = 3;
f = x (y''[x]) + α (y'[x]) + x (g (x, y[x])); (*DIfferential equation of the form (1.1)*)
y[0] = α; (*Where α is constant which is to be determined/assumption*)
y'[0] = 0; (*Initial condition*)
For[i = 0, i ≤ n, i++, Print[F_i = D[f, {x, i}]]]
A = Table[F_i == 0, {i, n}] /. x → 0
A // MatrixForm
Derivativevalue = Table[D[y[x], {x, i + 1}], {i, n}] /. x → 0
Derivativevalueatorigin = Flatten[Solve[A, Derivativevalue]]
Derivativevalueatorigin // MatrixForm
Derivativefinaloutput = Table[Derivativevalue[[i]] /. Derivativevalueatorigin[[i]], {i, 1, n}]
Sum[(D[y[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Final Taylor series solution y(x)*)
```

3.3. Coupled IVP

We consider the coupled system of equation with the help of Eq. (8) subject to the initial conditions in the following form:

$$u''(x) + \frac{\alpha}{x} u'(x) + g(x, u(x), v(x)) = 0, \qquad 0 < x < 1, \tag{19}$$

$$v''(x)+\frac{\beta}{x}v'(x)+h(x,u(x),v(x))=0, \qquad 0<x<1,$$
$$u(0)=c_1, v(0)=c_2, u'(0)=d_1, v'(0)=d_2, \tag{19-cont.}$$

where, c_1, c_2, d_1, d_2 are constants and $g(x,u(x),v(x))$ and $h(x,u(x),v(x))$ are arbitrary functions of x, u and v.

Now, we present a Mathematica code for Eq. (19) which gives Taylor series system of solutions up to n^{th} terms as a function of x.

```
n = 2;
α = 8;
β = 4;
f = x (u''[x]) + α ( u'[x]) +  x (h1 (x, u[x], v[x])); (*Differential equation of the form (1.1)*)
g = x (v''[x]) + β (v'[x]) + x (h2 (x, u[x], v[x])); (*Differential equation of the form (1.1)*)
u[0] = c1; (*Where c1 is constant/Initial condition*)
u'[0] = d1; (*Where d 1is constant/Initial condition*)
v[0] = c2; (*Where  c2 is constant/Initial condition*)
v'[0] = d2; (*Where d2 is constant/Initial condition*)
For[i = 0, i ≤ n, i++, Print[Fi = D[f, {x, i}]]]
For[i = 0, i ≤ n, i++, Print[Gi = D[g, {x, i}]]]
A = Table[Fi == 0, {i, n}] /. x → 0
B = Table[Gi == 0, {i, n}] /. x → 0
B // MatrixForm
A // MatrixForm
Derivativevalueu = Table[D[u[x], {x, i + 1}], {i, n}] /. x → 0
Derivativevaluev = Table[D[v[x], {x, i + 1}], {i, n}] /. x → 0
Differentialequation = Flatten[{A, B}]
Derivativevalue = Flatten[{Derivativevalueu, Derivativevaluev}]
Derivativevalueatorigin = Flatten[Solve[Differentialequation, Derivativevalue]]
Derivativevalueatorigin // MatrixForm
Derivativefinaloutput = Table[Derivativevalue[[i]] /. Derivativevalueatorigin[[i]], {i, 1, n}]
Sum[(D[u[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Final Taylor series solution u(x)*)
Sum[(D[v[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Final Taylor series solution v(x)*)
```

3.4. Coupled BVP

Here, we consider the following coupled system of equations subject to the boundary conditions in the following form:

$$u''(x)+\frac{\alpha}{x}u'(x)+g(x,u(x),v(x))=0, \qquad 0<x<1,$$
$$v''(x)+\frac{\beta}{x}v'(x)+h(x,u(x),v(x))=0, \qquad 0<x<1, \tag{20}$$
$$u(0)=0, v(0)=0, u'(1)=c_1, v'(1)=c_2,$$

where, c_1, c_2, are constants and $g(x,u(x),v(x))$ and $h(x,u(x),v(x))$ are arbitrary functions of x, u and v. Since the value of $u(0)$ and $v(0)$ are unknown, so we chose $u(0)=a$ and $v(0)=b$. In the following, we provide a Mathematica code for Eq. (20) which gives Taylor series system of solutions up to n^{th} terms as a function of x, a and b.

```
n = 2;
α = 8;
β = 4;
f = x (u''[x]) + α ( u'[x]) +  x (h1 (x, u[x], v[x])); (*Differential equation 1 of the form (1.1)*)
g = x (v''[x]) + β (v'[x]) + x (h2 (x, u[x], v[x])); (*Differential equation  2 of the form (1.1)*)
u[0] = a; (*Where a is constant which is to be determined/Assumptions*)
u'[0] = 0; (*Initial condition*)
v[0] = b; (*Where b is constant which is to be determined/Assumptions*)
v'[0] = 0; (*Initial condition*)
For[i = 0, i ≤ n, i++, Print[Fi = D[f, {x, i}]]]
For[i = 0, i ≤ n, i++, Print[Gi = D[g, {x, i}]]]
A = Table[Fi == 0, {i, n}] /. x → 0
B = Table[Gi == 0, {i, n}] /. x → 0
B // MatrixForm
```

```
A // MatrixForm
Derivativevalueu = Table[D[u[x], {x, i + 1}], {i, n}] /. x → 0
Derivativevaluev = Table[D[v[x], {x, i + 1}], {i, n}] /. x → 0
Differentialequation = Flatten[{A, B}]
Derivativevalue = Flatten[{Derivativevalueu, Derivativevaluev}]
Derivativevalueatorigin = Flatten[Solve[Differentialequation, Derivativevalue]]
Derivativevalueatorigin // MatrixForm
Derivativefinaloutput = Table[Derivativevalue[[i]] /. Derivativevalueatorigin[[i]], {i, 1, n}]
Sum[(D[u[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Taylor series solution u[x]*)
Sum[(D[v[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Taylor series solution v[x]*)
```

4. Taylor Series Solution for ODE

Here, we present few real-life problems as test examples to verify our code.

4.1. IVP

We consider some initial value problems stated below.

4.1.1. IVP 1

We consider equations (17) with $g(x,y) = y(x) - (6 + 12x + x^2 + x^3), c = 0, d = 0$ and $\alpha = 2$ which have an exact solution $x^2 + x^3$. By using the Mathematica code as in subsection 3.1, we get the Taylor series solution up to third terms which is $y^{Taylor}(x) \approx x^2 + x^3$. The accuracy of the method is better than variational iteration method (VIM) [48] (see Fig. 1).

4.1.2. IVP 2

Here we take the Lane-Emden type equations (17) $g(x,y) = 8\exp(y) + 4\exp(\frac{y}{2}), c = 0, d = 0$ and $\alpha = 2$ which have an exact solution $-2\log(1+x^2)$. With the help of Mathematica code given in subsection 3.1 we compute the Taylor series solution which is given as follows:

$$y^{Taylor}(x) \approx -2x^2, -2x^2 + x^4, \qquad -2x^2 + x^4 - \frac{2x^6}{3}, \cdots . \tag{21}$$

For $n \to \infty$, we get the closed form of the solution which is:

$$y(x) = y^{Taylor}(x) = -2x^2 + x^4 - \frac{2x^6}{3} - \cdots = -2\log(1+x^2). \tag{22}$$

We can also verify our obtained result by VIM [48].

4.1.3. IVP 3

We consider the Lane-Emden type equations (17) where $g(x,y) = -6y - 4y\log(y)$, $c = 1, d = 0$ and $\alpha = 2$ which have an exact solution $\exp(x^2)$. Now, by using Mathematica code given in subsection 3.1, we arrive at:

$$y^{Taylor}(x) \approx 1 + x^2, 1 + x^2 + \frac{x^4}{2}, \qquad 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}, \cdots . \tag{23}$$

For $n \to \infty$, we get the closed form of the solution which is $\exp(x^2)$. Also, we see that each of these approximations are same as approximations calculated by VIM [48].

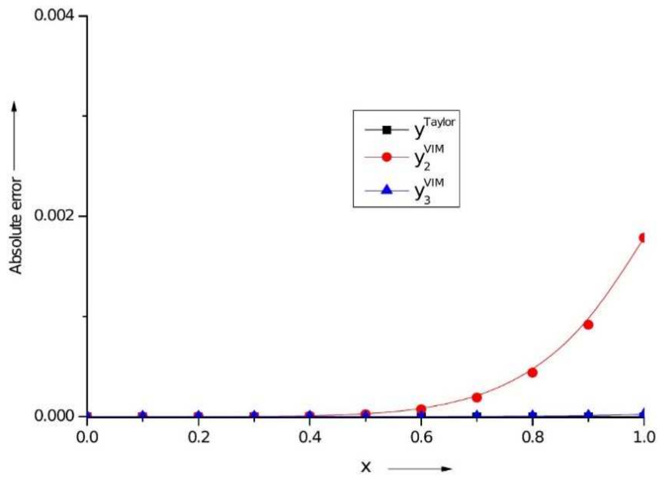

Fig. 1. Comparison between Taylor series method and VIM.

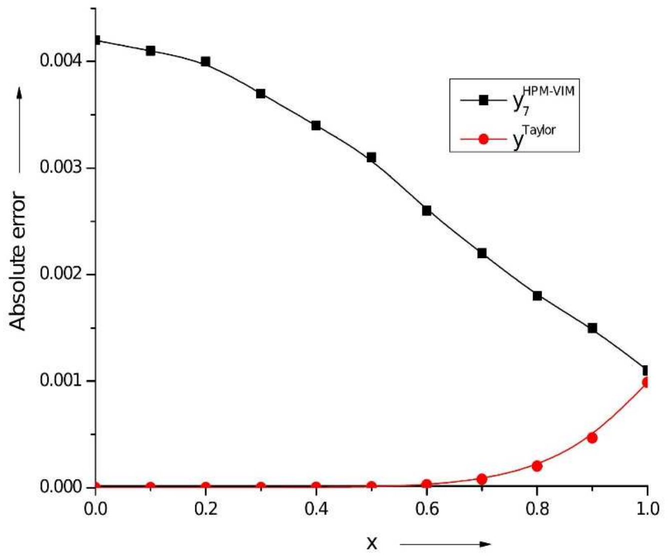

Fig. 2. Comparison between absolute error of Taylor series method and HPM-VIM.

4.2. BVP

He [46] applied the Taylor series method on the Bratu type boundary value problem which is nonlinear but regular and computed the approximate solution. He also showed that this method is quite powerful than the iterative method VIM. Here, we consider a few highly non-linear singular boundary value problems of Lane-Emden type and verify the Mathematica codes given in subsection 3.2.

4.2.1. BVP 1

First, we consider linear second order singular boundary value problem (18) with $g(x,y)=\exp(-y)$, $\beta_1=2,\ \beta_2=1, d=0$ and $\alpha=2$. The exact solution is not known. Now, by using the code and Mathematica, we have:

$$y^{Taylor}(x) \approx\ c-\frac{\exp(-3c)x^6}{1890}\ -\frac{1}{120}\exp(-2c)x^4-\frac{1}{6}\exp(-c)x^2, \tag{24}$$

where $c=y(0)$. By using the boundary condition $2y(1)+y'(1)=0$ we have $c=0.269977$. We see that the Taylor series solution gives better accuracy than HPM-VIM [49] combined iterative approximation. Below we provide an absolute error graph (see Fig. 2) for comparison between these two methods.

4.2.2. BVP 2

Now, we consider a model based on equilibrium isothermal gas sphere arising in astronomy of the form of Eq. (18) where $g(x,y)=y^5$, $\beta_1=0,\ \beta_2=1, d=\sqrt{3}/2$ and $\alpha=2$. The exact solution of BVP is $1/\sqrt{(1+x^2/3)}$. By using the Mathematica code given in subsection 3.2 we get second order and 12^{th} order Taylor series approximations, given as follows:

$$y^{Taylor}(x) \approx\ \ c-\frac{c^5x^2}{6}\ , \tag{25}$$

$$y^{Taylor}(x) \approx\ \ c+\frac{77c^{25}x^{12}}{248832}-\frac{7c^{21}x^{10}}{6912}+\frac{35c^{17}x^8}{10368}-\frac{5c^{13}x^6}{432}+\frac{c^9x^4}{24}-\frac{c^5x^2}{6}\ , \tag{26}$$

Now, by using the boundary condition $y(1)=\sqrt{3}/2$, we have $c=0.999832$. Therefore, for $n=12$, the Taylor series approximation is:

$$y^{Taylor}(x) \approx\ \ 0.000308\ x^{12}-0.001009x^{10}+0.003366x^8-0.0115489x^6+0.04160x^4-0.166527x^2+0.999832. \tag{27}$$

Now, we compare our solution (27) with the solution computed by VIM which is:

$$y^{VIM}(x) \approx\ \ 7\times10^{-12}\ x^{12}-0.00003x^{10}+0.000577x^8-0.00609x^6+0.039355x^4-0.161465x^2+0.993678. \tag{28}$$

From Fig. 3, we observe that Taylor series solution gives better accuracy than VIM solution [50].

4.2.3. BVP 3

Here, we take the equation of shallow membrane cap of the form of Eq. (18) with $g(x,y)=-\frac{1}{2}+\frac{1}{8y^2}, d=1$ and $\alpha=3$. It has no exact solution. Again, by using the Mathematica code given in subsection 3.2 we get Taylor series approximation up to second term as follows:

$$y^{Taylor}(x) \approx\ \ \ c+\frac{(4c^2-1)x^2}{64c^2}, \tag{29}$$

where, $c=y(0)$. By using the boundary condition $y(1)=0$, we have $c=0.954645$. Hence, the Taylor series approximation is:

$$y^{Taylor}(x) \approx\ \ \ 0.045355x^2+0.654645. \tag{30}$$

We compare our computed approximation with the VIM [50] approximation, which is:

$$y_1^{VIM}(x) \approx\ \ \ 0.0483493x^2+0.951651. \tag{31}$$

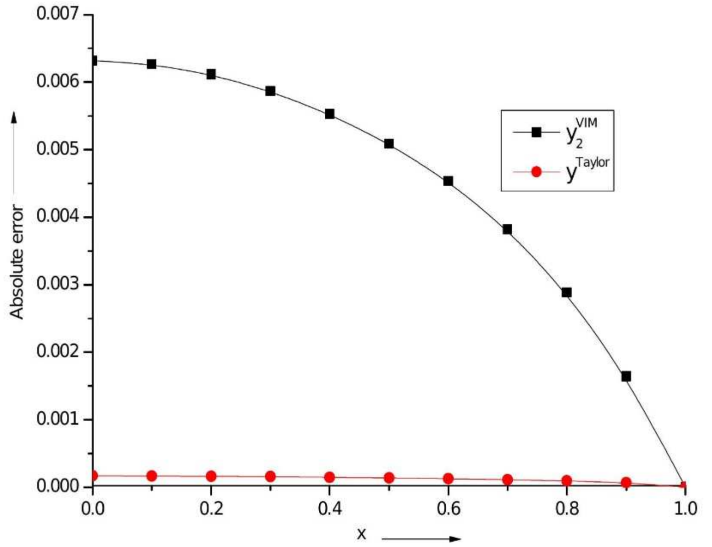

Fig. 3. Comparison between absolute error of Taylor series method and VIM.

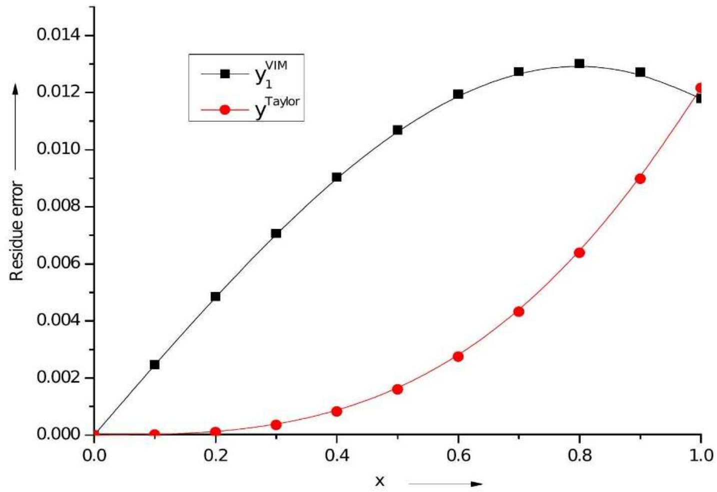

Fig. 4. Comparison between absolute error of Taylor series method and VIM.

We saw that the Taylor series solution gives better approximation than VIM (see Fig. 4).

4.3. Coupled IVP

He et al. [44] computed closed form solution of coupled IVP by using Taylor series method. Our main aim in this section is to verify our developed Mathematica code as in subsection 3.3 by considering different highly non-linear real-life problems. To achieve our goal first we take a real-life example which is described in [1]. We also found closed form solution as given in [1]. Now, we consider two real life problems.

4.3.1. *Coupled IVP 1*

Wazwaz et al. [51] considered the system of Lane-Emden type equations of the form of Eq. (19) with $g(x,u,v) = 8\,(\exp(u) + 2\exp(-\frac{v}{2}))$, $h(x,u,v) = -8(\exp(-v) + \exp(\frac{u}{2})$, $\beta = 3$, $c_1 = 0, c_2 = 0, d_1 = 0, d_2 = 0$ and $\alpha = 5$. Exact solution of this coupled IVP is $((u(x), v(x)) = (-2\log(1+x^2), 2\log(1+x^2))$. They used the Adomian's decomposition method (ADM) to find the approximate system of solutions. Now, by using Taylor series method, we arrive at:

$$u^{Taylor}(x) \approx \frac{x^{12}}{3} - \frac{2x^{10}}{5} + \frac{x^8}{2} - \frac{2x^6}{3} + x^4 - 2x^2, \tag{32}$$

$$v^{Taylor}(x) \approx -\frac{x^{12}}{3} + \frac{2x^{10}}{5} - \frac{x^8}{2} + \frac{2x^6}{3} - x^4 + 2x^2. \tag{33}$$

Therefore, for $n \to \infty$ we get the closed form system of solutions $((u(x), v(x)) = (-2\log(1+x^2), 2\log(1+x^2))$. which are same as computed by ADM.

4.3.2. *Coupled IVP 2*

Here, we consider the system of equation (19) [51] with $g(x,u,v) = 18u - 4u\log(v)$, $h(x,u,v) = 4v\log(u) - 10v$, $\beta = 4$, $c_1 = 1, c_2 = 1, d_1 = 0, d_2 = 0$ and $\alpha = 8$. Exact solution of this coupled IVP is $((u(x), v(x)) = (\exp(-x^2), \exp(x^2))$. With the help of Mathematica code, we get Taylor series solution as follows:

$$u^{Taylor}(x) \approx \frac{x^{12}}{720} - \frac{x^{10}}{120} + \frac{x^8}{24} - \frac{x^6}{6} + \frac{x^4}{2} - x^2 + 1, \tag{34}$$

$$v^{Taylor}(x) \approx \frac{x^{12}}{720} + \frac{x^{10}}{120} + \frac{x^8}{24} + \frac{x^6}{6} + \frac{x^4}{2} + x^2 + 1. \tag{35}$$

When $n \to \infty$ we have the exact system of solutions $(\exp(-x^2), \exp(x^2))$.

4.4. Coupled BVP

In the following, we present few numerical examples related to system of boundary value problem.

4.4.1. *Coupled BVP 1*

Consider the system of differential equation (20) $g(x,u,v) = 18u - 4u\log(v)$, $h(x,u,v) = 4v\log(u) - 10v$, $\beta = 4$, $c_1 = \frac{1}{\exp(1)}, c_2 = \exp(1)$, and $\alpha = 8$. Exact solution of this coupled IVP is $((u(x), v(x)) = (\exp(-x^2), \exp(x^2))$. Now using algorithm described in subsection 3.4, we compute the approximate Taylor series solution. For $n = 5$:

$$u^{Taylor}(x) \approx -\left(\frac{x^4}{990}\right)(-20a(\log(b))\ (\log(b))\ + 180a \log(b) - 495a + 36a \log(a)\ ) + \frac{1}{9}x^2(2a\log(b) - 9a) + a, \tag{36}$$

$$v^{Taylor}(x) \approx \left(\frac{x^4}{630}\right)(-36b(\log(a))\ (\log(a))\ + 180b \log(a) + 315b + 20b \log(b)\ ) - \frac{1}{5}x^2(2b\log(a) - 5b) + b, \tag{37}$$

where $a = u(0)$ and $b = v(0)$. For $n = 10$ we have computed the values of a and b by using boundary conditions $u(1) = \frac{1}{\exp(1)}$ and $v(1) = \exp(1)$ which are $a = 1.00299$ and $b = 1.00156$. Therefore, for $n = 10$, the system of Taylor series solutions is:

$$u^{Taylor}(x) \approx -0.00833149\mathrm{x}^{10} + 0.0417031\ \mathrm{x}^8 - 0.166939\ \mathrm{x}^6 + 0.5011\ \mathrm{x}^4 - 1.00264\ \mathrm{x}^2 + 1.00299, \tag{38}$$

$$v^{Taylor}(x) \approx 1.00156 + 1.00037\ \mathrm{x}^2\ +\ 0.499878\ \mathrm{x}^4\ +\ 0.166539\ \mathrm{x}^6\ +\ 0.0416179\ \mathrm{x}^8 + 0.00832063\ \mathrm{x}^{10}. \tag{39}$$

Here in Fig. 5, we plot absolute error of Taylor series solution of Eq. (20).

4.4.2. *Coupled BVP 2*

We consider the system of differential equation (20) with $g(x,u,v) = 8\ (\exp(u-1) + 2\exp(-\frac{v-1}{2}))$, $h(x,u,v) = -8(\exp(-(v-1)) + \exp(\frac{u-1}{2})$, $\beta = 3$, $c_1 = 1 - 2\log(2), c_2 = 1 + 2\log(2)$ and $\alpha = 5$. Exact solution of this coupled BVP is $((u(x), v(x)) = (1 - 2\log(1+x^2), 1 + 2\log(1+x^2))$. Now, using algorithm developed in subsection 3.4 for $n = 5$, we have the following Taylor series approximation:

$$u^{Taylor}(x) \approx \left(\frac{x^4}{12}\right)\left(3\exp\left(\frac{a}{2} - \frac{b}{2}\right) + 4\exp\left(a - \frac{b}{2} - \frac{1}{2}\right) + 2\exp(2a-2) + 3\exp\left(\frac{3}{2} - \frac{3b}{2}\right)\right) + \frac{2}{3}x^2\left(\exp(a-1) + 2\exp\left(\frac{1}{2} - \frac{b}{2}\right)\right) + a, \tag{40}$$

$$v^{Taylor}(x) \approx \left(\frac{x^4}{9}\right)\left(-3\exp\left(\frac{a}{2} - b + \frac{1}{2}\right) - 2\exp\left(\frac{a}{2} - \frac{b}{2}\right) - \exp\left(\frac{3a}{2} - \frac{3}{2}\right) - 3\exp(2-2b)\right) - \exp\left(-b - \frac{1}{2}\right)x^2\left(\exp\left(\frac{a}{2} + b\right) + \exp\left(\frac{3}{2}\right)\right) + b, \tag{41}$$

where $a = u(0)$ and $b = v(0)$. Now from $u(1) = 1 - 2\log(2)$ and $v(1) = 1 + 2\log(2)$, we have two nonlinear system of equations. Therefore, by using Newton Raphson method, we get the values of a and b which are given by $a = 1$ and $b = 1$. Hence, for $n = 5$ Taylor series solution of this coupled system are:

$$u^{Taylor}(x) \approx \mathrm{x}^4 - 2\ \mathrm{x}^2 + 1, \tag{42}$$

$$v^{Taylor}(x) \approx -\mathrm{x}^4 + 2\ \mathrm{x}^2 + 1. \tag{43}$$

And for $n \to \infty$, we get exact system of solutions.

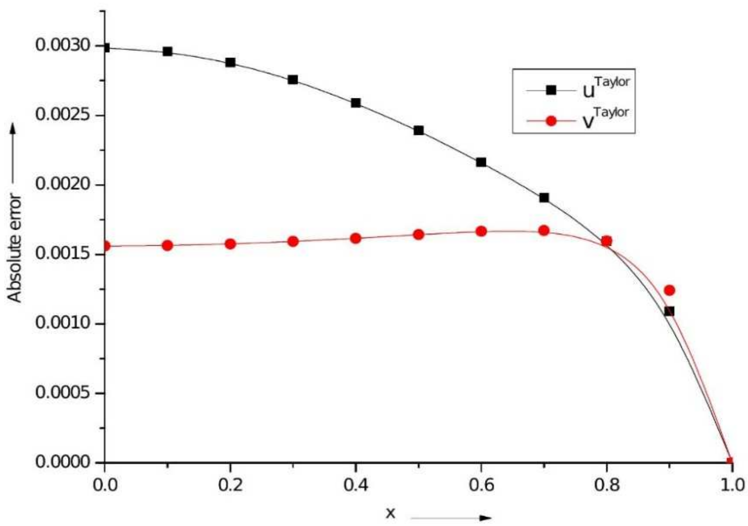

Fig. 5. Absolute error of Taylor series method.

5. Taylor Series Solution for PDE

In this section, we derive analytical solution for three sets of problems. First is two nonlinear PDEs with initial condition and the second one is system of nonlinear PDEs subject to given initial conditions.

5.1. Burgers' equation

We consider the following class of Burgers' equations:

$$\begin{aligned} &\text{PDE:} \qquad u_t + uu_x = \mu u_{xx}, \qquad 0 \le x \le 1, t > 0 \\ &\text{Initial Condition:} \qquad u(x,0) = 2\,x, \\ &\text{Exact Solution:} \qquad u(x,t) = \frac{2x}{1+2t}. \end{aligned} \tag{44}$$

Using Eq. (44), we get:

$$u(0{,}0) = 0, u_x(0{,}0) = 2, u_{xx}(0{,}0) = 0, u_t(0{,}0) = 0. \tag{45}$$

Differentiating with respect to x and t, respectively, we get:

$$u_{xt} + (u_x)^2 + uu_{xx} = \mu u_{xxx}, \tag{46}$$

$$u_{tt} + u_t u_x + uu_{tx} = \mu u_{txx}. \tag{47}$$

Differentiating Eq. (46) with respect to x and t, and Eq. (47) with respect to t, we have:

$$u_{xxt} + 2(u_x)u_{xx} + uu_{xxx} + u_x u_{xx} = \mu u_{xxxx}, \tag{48}$$

$$u_{xtt} + 2u_{tx}u_x + u_t u_{xx} + uu_{txx} = \mu u_{txxx}. \tag{49}$$

$$u_{ttt} + u_{tt}u_x + u_t u_{tx} + u_t u_{tx} + uu_{ttx} = \mu u_{ttxx}. \tag{50}$$

Differentiating Eq. (48) with respect to x, we get:

$$u_{xxxt} + 2u_x u_{xxx} + 3(u_{xx})^2 + uu_{xxxx} + 2u_x u_{xxx} = \mu u_{xxxxx}. \tag{51}$$

Therefore, we get:

$$u_{xxxt} + 2u_x u_{xxx} + 3(u_{xx})^2 + uu_{xxxx} + 2u_x u_{xxx} = \mu u_{xxxxx}. \tag{52}$$

$$u_{xt}(0{,}0) = -4, u_{tt}(0{,}0) = 0, u_{xxx}(0{,}0) = 0, u_{xxt}(0{,}0) = 0, u_{xtt}(0{,}0) = 16, u_{ttt}(0{,}0) = 0. \tag{53}$$

Taylor series expansion of $u(x,t)$ around the point $(0{,}0)$ can be written as:

$$u(x,t) = u(0{,}0) + \frac{1}{1!}(u_x(0{,}0)x + u_t(0{,}0)t) + \frac{1}{2!}(u_{xx}(0{,}0)x^2 + u_{tt}(0{,}0)t^2 + 2u_{xt}(0{,}0)xt) + \cdots \tag{54}$$

Substituting the values of $u(0{,}0)$, $u_x(0{,}0)$ and all other values in Eq. (54), we get:

$$u(x,t) = 2x - 4xt + 8xt^2 + \cdots = \frac{2x}{1+2t}. \tag{55}$$

which is same as computed in [38, 39].

5.2. KDV equation

We consider KDV equation:

$$u_t - 6uu_x + u_{xxx} = 0, 0 < x < 1 \text{ and } t > 0, \tag{56}$$

$$u(x,0) = -\frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx}{2}\right). \tag{57}$$

The exact solution of KDV equation is $-\frac{k^2}{2}\operatorname{sech}^2\left(\frac{k}{2}(x-k^2t)\right)$.
By using equations (56) and (57), we have:

$$u\,(0{,}0) = -\frac{k^2}{2}, u_x(0{,}0) = 0, u_{xx}(0{,}0) = \frac{k^4}{4}, u_{xxx}(0{,}0) = 0, u_t(0{,}0) = 0. \tag{58}$$

Differentiating equation (56) with x and t, we have:

$$u_{xt} - 6(u_x)^2 - 6uu_{xx} + u_{xxxx} = 0, \tag{59}$$

$$u_{tt} - 6u_t u_x - 6uu_{tx} + u_{txxx} = 0. \tag{60}$$

To find the value of u_{txxx}, we differentiate equation (59) with respect to x and we get:

$$u_{xxt} - 6uu_{xxx} - 6u_x u_{xx} - 12u_x u_{xx} + u_{xxxxx} = 0, \tag{61}$$

$$u_{xxxt} - 18(u_{xx})^2 - 24u_x u_{xxx} - 6uu_{xxxx} + u_{xxxxxx} = 0. \tag{62}$$

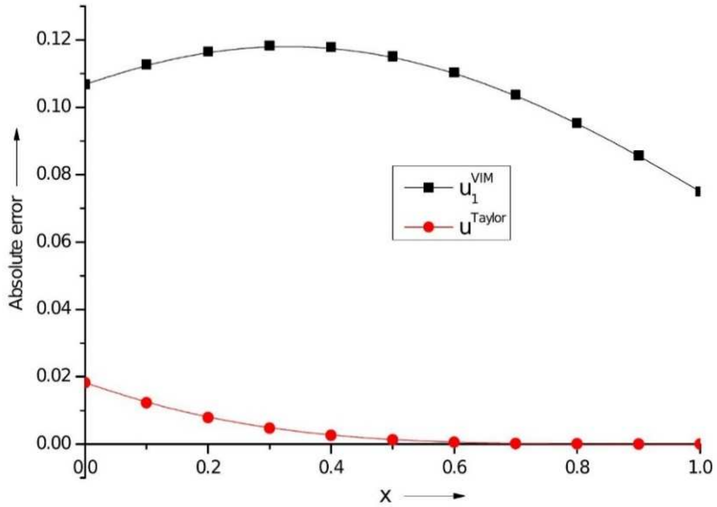

Fig. 6. Absolute error of $u(x,t)$ for $t = 1$ for KDV equation.

By using equations (60), (61) and (62), we have:

$$u_{xxxx}(0,0) = -\frac{k^6}{2}, u_{xxxxx}(0,0) = 0, u_{xxxxxx}(0,0) = 17\frac{k^8}{8}, u_{xt}(0,0) = -\frac{k^6}{4}, u_{tt}(0,0) = \frac{k^8}{4}. \quad (63)$$

Therefore, first order Taylor series solution is:

$$u^{Taylor}(x,t) \approx -\frac{k^2}{2}\left(1 - \left(\frac{k}{2}\,(x - k^2 t)\right)^2\right). \quad (64)$$

The KDV equation (56) and (57) have numerically solved by VIM in [2]. First approximation of VIM solution is given by:

$$u_1^{VIM}(x,t) \approx -\frac{k^2}{2}\mathrm{sech}^2\left(\frac{k}{2}x\right) - \frac{k^5}{2}\mathrm{sech}^2\left(\frac{k}{2}x\right)\tanh^2\left(\frac{kx}{2}\right)t\,. \quad (65)$$

From Fig. 6, we see that Taylor series solution provide better accuracy than VIM approximation.

5.3. System of nonlinear PDE

We consider the following system of nonlinear PDEs with given initial conditions:

$$u_t + uu_x + vu_y = \tfrac{1}{Re}(u_{xx} + u_{yy}), 0 \le x \le 1 \text{ and } t > 0, \quad (66)$$

$$v_t + uv_x + vv_y = \tfrac{1}{Re}(v_{xx} + v_{yy}), 0 \le x \le 1 \text{ and } t > 0, \quad (67)$$

$$\text{Initial Condition: } u(x,y,0) = x + y, v(x,y,0) = x - y, \quad (68)$$

$$\text{Exact Solution: } u(x,y,t) = \tfrac{x+y-2xt}{1-2t^2}, \quad (69)$$

$$v\,(x,y,t) = \frac{x - y - 2yt}{1 - 2t^2}. \quad (70)$$

Here Re is known as Reynold's number which is related to viscous property of fluid. Now:

$$u_x(x,y,0) = 1, u_y(x,y,0) = 1, u_{xx}(x,y,0) = 0, u_{yy}(x,y,0) = 0, \quad (71)$$

$$v_x(x,y,0) = 1, v_y(x,y,0) = -1, v_{xx}(x,y,0) = 0, v_{yy}(x,y,0) = 0, \quad (72)$$

$$u(0,0,0) = 0, v(0,0,0) = 0, u\_t(0,0,0) = 0. \quad (73)$$

Differentiating Eq. (66) with respect to x and t, we have:

$$u_{xt} = -u_x^2 - u\,u_{xx} - v_x u_y - v\,u_{xy} + \frac{1}{Re}\,(u_{xxx} + u_{xyy}), \quad (74)$$

$$u_{tt} = -u_t u_x - u\,u_{tx} - v_t\,u_y - v\,u_{ty} + \frac{1}{Re}\,(u_{txx} + u_{tyy}), \quad (75)$$

$$u_{yt} = -u_y u_x - u\,u_{yx} - v_y u_y - v\,u_{yy} + \frac{1}{Re}\,(u_{yxx} + u_{yyy}), \quad (76)$$

$$u_{xxt} = -2\,u_x u_{xx} - u_x u_{xx} - u\,u_{xxx} - v_{xx}u_y - v_x u_{xxy} + \frac{1}{Re}\,(u_{xxxx} + u_{xxyy}), \quad (77)$$

$$u_{yyt} = -u_{yy}\ u_x - u_y u_{yx} - u_y\ u_{yx} - u\ u_{yyx} - v_{yy}\ u_y - v_y\ u_{yy} -\ v_y\ u_{yy} -\ v\ u_{yyy} + \frac{1}{Re}(u_{yyxx} + u_{yyyy}), \tag{78}$$

$$u_{xtt} = -2u_x\ \ u_{tx} - u_t\ u_{xx} - u\ u_{txx} - v_{tx}\ u_y - v_x\ u_{ty}\ - v_t\ u_{xy} - v\ u_{txy} + \frac{1}{Re}\ (u_{txxx} + u_{txyy}), \tag{79}$$

$$u_{xyt} = -u_{xy}\ u_x - u_y\ u_{xx} - u_x\ u_{yx} - u\ u_{xyx} - v_{xy}\ u_y - v_y\ u_{xy} - v_x\ u_{yy} - v\ u_{xyy} + \frac{1}{Re}\ (u_{xyxx} + u_{xyyy}), \tag{80}$$

$$u_{ytt} = -u_{yt}u_x - u_t u_{yx} - u_y\ u_{tx} - u\ u_{ytx} - v_{yt}u_y - v_t u_{yy} - v_y u_{ty} - v\ u_{yty} + \frac{1}{Re}(u_{ytxx} + u_{ytyy}). \tag{81}$$

Hence, we deduce the following:

$$u_{xt}(0{,}0{,}0) = -2, u_{tt}(0{,}0{,}0) = 0, u_{yt}(0{,}0{,}0) = 0, \tag{82}$$

$$u_{xxt}(0{,}0{,}0) = 0, u_{yyt}(0{,}0{,}0) = 0, \tag{83}$$

$$u_{xyt}(0{,}0{,}0) = 0, u_{xtt}(0{,}0{,}0) = 4, u_{ytt}(0{,}0{,}0) = 4, \tag{84}$$

$$u_{xxxt}(0{,}0{,}0) = 0, u_{yyyt}(0{,}0{,}0) = 0, u_{yyxt}(0{,}0{,}0) = 0. \tag{85}$$

Similarly, for $v(x,y,t)$, we have:

$$v_{xt}(0{,}0{,}0) = -2, v_{tt}(0{,}0{,}0) = 0, v_{yt}(0{,}0{,}0) = 0, \tag{86}$$

$$v_{xxt}(0{,}0{,}0) = 0, v_{yyt}(0{,}0{,}0) = 0, \tag{87}$$

$$v_{xyt}(0{,}0{,}0) = 0, v_{xtt}(0{,}0{,}0) = 4, v_{ytt}(0{,}0{,}0) = 4, \tag{88}$$

$$v_{xxxt}(0{,}0{,}0) = 0, v_{yyyt}(0{,}0{,}0) = 0, v_{yyxt}(0{,}0{,}0) = 0. \tag{89}$$

For the sake of brevity, we are not providing calculations further. Taylor series expansion of $u(x,y,t)$ and $v(x,y,t)$ around the point $(0{,}0{,}0)$ can be written as:

$$\begin{aligned} u(x,y,t) &= u(0{,}0{,}0) + \frac{1}{1!}(u_x(0{,}0{,}0)x + u_y(0{,}0{,}0)y +\ u_t(0{,}0{,}0)t) \\ &+ \frac{1}{2!}(\ u_{xx}(0{,}0{,}0)x^2 + u_{yy}(0{,}0{,}0)y^2 + u_{tt}(0{,}0{,}0)t^2 +\ 2\ u_{xy}(0{,}0{,}0)x\ y + 2\ u_{xt}(0{,}0{,}0)x\ t + 2\ u_{yt}(0{,}0{,}0)y\ t) + \cdots, \end{aligned} \tag{90}$$

$$\begin{aligned} v(x,y,t) &= v(0{,}0{,}0) + \frac{1}{1!}(v_x(0{,}0{,}0)x + v_y(0{,}0{,}0)y +\ v_t(0{,}0{,}0)t) \\ &+ \frac{1}{2!}(\ v_{xx}(0{,}0{,}0)x^2 + v_{yy}(0{,}0{,}0)y^2 + v_{tt}(0{,}0{,}0)t^2 +\ 2\ v_{xy}(0{,}0{,}0)x\ y + 2\ v_{xt}(0{,}0{,}0)x\ t + 2\ v_{yt}(0{,}0{,}0)y\ t) + \cdots\ . \end{aligned} \tag{91}$$

Substituting the values of u and v and their derivatives at $(0{,}0{,}0)$ in Eqs. (90) and (91), we arrive at:

$$u(x,y,t) = x + y - 2xt + 2xt^2 + 2yt^2 + \cdots = \frac{x + y - 2xt}{1 - 2t^2}, \tag{92}$$

$$v(x,y,t) = x - y - 2yt + 2xt^2 - 2yt^2 + \cdots = \frac{x - y - 2yt}{1 - 2t^2}. \tag{93}$$

Remark: These derivations are based on assumption that mixed derivatives u_{xy} and u_{yx} and all other higher order derivatives all are same.

6. Conclusions

In this paper, we extended the work presented by He et al. [45] to various real-life problems which are highly nonlinear in nature. We successfully developed a few Mathematica codes to solve a class of singular non-linear ODEs subject to initial conditions and boundary conditions. We developed the codes for the system of non-linear singular ODEs and solved them successfully too. We also extended this approach to PDEs. This approach can further be extended to a different class of problems that do not have exact solutions. Finally, we concluded that this simple technique is very useful for engineering, chemical, and physical sciences.

Author Contributions

Basic idea of the paper was conceived by J.H. He, L. Verma, B. Pandit and A.K. Verma. B. Pandit has written programs in Mathematica. R.P. Agarwal and A.K. Verma worked and suggested on examples related to singular BVPs. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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