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Research Paper

Stability Analysis of Gyroscopic Systems with Delay under Synchronous and Asynchronous Switching

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Abstract. With the aid of the decomposition method and the Lyapunov direct method, stability of linear gyroscopic systems with switching and a constant delay in positional forces is investigated. The cases of synchronous and asynchronous switching are studied. The efficiency of the application of the Razumikhin approach and Lyapunov—Krasovskii functionals for the stability analysis of such systems is compared. The results of a numerical simulation are presented to illustrate the obtained theoretical conclusions.

Keywords: Gyroscopic systems, asymptotic stability, switching, delay, decomposition.

1. Introduction

In numerous applications, motions of gyroscopic systems are modeled by the second-order linear differential equations of high dimension with a large positive parameter which can be interpreted as the kinetic moment of fast rotating rotors containing in these systems [1-6]. The high dimension significantly complicates stability analysis of such systems. It is well-known (see [3, 7-9]), that an effective tool for the investigation of stability of large-scale systems is the decomposition method. The application of this method to gyroscopic systems is based on the precessional theory [2, 3]: an initial second-order system is decomposed into two first-order subsystems (precessional and nutational ones). In [2, 3], with the aid of the Lyapunov first method, it was proved that, for sufficiently large values of the parameter, the asymptotic stability of the precessional and nutation subsystems implies the asymptotic stability of the initial system. In [10], a similar result was obtained for gyroscopic systems with constant delay in positional forces.

Another approach for justification of the application of the precessional theory was developed in [11-15]. It is based on the Lyapunov functions method. Therefore, unlike to the first approach, the second one was effectively used to derive stability conditions for time-varying and some types of nonlinear systems.

In the present contribution, we study linear mechanical systems with dissipative, gyroscopic and switched positional forces. It is assumed that the considered systems contain a constant delay in the positional forces and a large positive parameter as a multiplier at the matrix of gyroscopic forces. Both cases of synchronous and asynchronous switching are investigated. It is worth mentioning that asynchronous switching occurs in various applications due to presence of a communication channel between the information about the switching law of the plant and that of the controller, or where some time is required to identify an active subsystem and apply the matched controller (see [16-19]).

Our objective is to obtain the conditions guaranteeing asymptotic stability under arbitrary admissible switching signal. To solve this problem, we use the decomposition method in the form proposed in [11-15] and the Lyapunov direct method. The comparison of the asymptotic stability conditions derived with the aid of the Razumikhin approach [20] and Lyapunov-Krasovskii functionals [21] is provided.

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2. Problem Formulation

Let two linear gyroscopic systems with delay and synchronous

$$A\ddot{q}(t)+(B+hG)\dot{q}(t)+C_{\sigma(t)}q(t)+D_{\sigma(t)}q(t-\tau)=0 \tag{1}$$

and asynchronous

$$A\ddot{q}(t)+(B+hG)\dot{q}(t)+C_{\sigma(t)}q(t)+D_{\sigma(t-\tau)}q(t-\tau)=0 \tag{2}$$

switching be given. Here $q(t),\dot{q}(t)\in R^n$ are vectors of generalized coordinates and generalized velocities, respectively, $\sigma:[-\tau,+\infty)\mapsto\{1,\ldots,N\}$ is a piecewise constant right-continuous function defining switching signal, N is the number of operating modes for the systems, A,B,G,C_s,D_s are constant matrices, $s=1,\ldots,N$, h is a positive parameter, τ is a constant positive delay. It is assumed that the matrices A,B are symmetric and positive definite, while the matrix G is skew-symmetric and nonsingular. Thus, we consider mechanical systems with dissipative, gyroscopic and switched .positional forces. The nonsingularity of G implies that n is an even number. In addition, assume that admissible switching signals are non Zeno, i.e., on every bounded time interval, the function $\sigma(t)$ admits only finite set of discontinuity instants [22].

Let initial functions for solutions of (1) and (2) belong to the space $C^1([-\tau,0],R^n)$ of continuously differentiable functions $\varphi(\xi):[-\tau,0]\mapsto R^n$ with the norm $\|\varphi\|_\tau=\max_{\xi\in[-\tau,0]}(\|\varphi(\xi)\|+\|\dot{\varphi}(\xi)\|)$ and $\|\bullet\|$ be the Euclidean norm of a vector. Denote by q_t the restriction of a solution $q(t)$ to the segment $[t-\tau,t]$ [20].

The forces $-D_{\sigma(t)}q(t-\tau)$ and $-D_{\sigma(t-\tau)}q(t-\tau)$ can be obtained as a result of application of controls with delay in the feedback law [21-23]. In the case of synchronous switching, the control is common for all operating modes ($-D_{\sigma(t)}q(t-\tau)=L_{\sigma(t)}u(t-\tau)$ and $u(t)=Kq(t)$), whereas, in the case of asynchronous switching, the control is mode-depended ($-D_{\sigma(t-\tau)}q(t-\tau)=Lu(t-\tau)$ and $u(t)=K_{\sigma(t)}q(t)$).

We will look for conditions ensuring asymptotic stability of (1) and (2) for any admissible switching law. To derive such conditions, we will use the decomposition method and the Lyapunov direct method. The efficiency of the application of the Razumikhin approach and Lyapunov—Krasovskii functionals for the stability analysis of the decomposed systems will be compared.

3. Decomposition

According to the precessional theory [2, 3], construct for (1) the nutational subsystem

$$A\dot{y}(t)+(B+hG)y(t)=0 \tag{3}$$

and the family of precessional subsystems

$$hG\dot{x}(t)+(C_s+D_s)x(t)=0\,, \qquad s=1,\ldots,N\,. \tag{4}$$

Similarly, for (2), we consider the subsystem (3) and the family

$$hG\dot{x}(t)+(C_s+D_r)x(t)=0\,, \qquad r,s=1,\ldots,N\,. \tag{5}$$

Remark 1. From the conditions imposed on the matrices A,B,G it follows that the system (3) is asymptotically stable.

Assumption 1. Every subsystem from the family (4) is asymptotically stable, and there exists a constant symmetric positive definite matrix P such that the matrices

$$PG^{-1}(C_s+D_s)-(C_s+D_s)^{T}G^{-1}P\,, \qquad s=1,\ldots,N\,,$$

are positive definite.

Assumption 2. Every subsystem from the family (5) is asymptotically stable, and there exists a constant symmetric positive definite matrix $\tilde{P}$ such that the matrices

$$\tilde{P}G^{-1}(C_s+D_r)-(C_s+D_r)^{T}G^{-1}\tilde{P}\,, \qquad r,s=1,\ldots,N\,,$$

are positive definite.

Remark 2. It is obvious that if Assumption 2 is fulfilled, then Assumption 1 is fulfilled, as well.

Remark 3. Under Assumption 1 (Assumption 2), the family (4) (family (5)) admits a common Lyapunov function of the form $V(x)=x^TPx$ ($V(x)=x^T\tilde{P}x$) satisfying the conditions of the Lyapunov asymptotic stability theorem.

Next, with the aid of the substitution

$$y(t)=\dot{q}(t)\,, \qquad (B+hG)x(t)=A\dot{q}(t)+(B+hG)q(t)\,, \tag{6}$$

transform (1) to the system

$$\begin{aligned} hG\dot{x}(t)&=-C_{\sigma(t)}x(t)-D_{\sigma(t)}x(t-\tau)+G(B+hG)^{-1}BG^{-1}\left(C_{\sigma(t)}x(t)+D_{\sigma(t)}x(t-\tau)\right)\\ &\quad+hG(B+hG)^{-1}\left(C_{\sigma(t)}(B+hG)^{-1}Ay(t)+D_{\sigma(t)}(B+hG)^{-1}Ay(t-\tau)\right),\\ A\dot{y}(t)&=-(B+hG)y(t)-C_{\sigma(t)}x(t)-D_{\sigma(t)}x(t-\tau)+C_{\sigma(t)}(B+hG)^{-1}Ay(t)+D_{\sigma(t)}(B+hG)^{-1}Ay(t-\tau). \end{aligned} \tag{7}$$

It is worth noting that (7) can be treated as a complex system describing at each time instant the interaction of the subsystem (3) and a subsystem from the family (4).

In a similar way, we transform the system (2) to the following one:

$$\begin{aligned} hG\dot{x}(t) &= -C_{\sigma(t)}x(t) - D_{\sigma(t-\tau)}x(t-\tau) + G(B+hG)^{-1}BG^{-1}\left(C_{\sigma(t)}x(t) + D_{\sigma(t-\tau)}x(t-\tau)\right) \\ &\quad + hG(B+hG)^{-1}\left(C_{\sigma(t)}(B+hG)^{-1}Ay(t) + D_{\sigma(t-\tau)}(B+hG)^{-1}Ay(t-\tau)\right), \\ A\dot{y}(t) &= -(B+hG)y(t) - C_{\sigma(t)}x(t) - D_{\sigma(t-\tau)}x(t-\tau) + C_{\sigma(t)}(B+hG)^{-1}Ay(t) + D_{\sigma(t-\tau)}(B+hG)^{-1}Ay(t-\tau). \end{aligned} \tag{8}$$

Remark 4. The substitution (6) does not change the stability property. Therefore, the system (1) (the system (2)) is asymptotically stable if and only if the same property is valid for (7) (for (8)).

For the stability analysis of (7) and (8), we will use both the Razumikhin approach and the Lyapunov—Krasovskii approach.

4. Application of the Razumikhin Approach

First, consider the synchronous case. In [24], the following theorem was proved.

Theorem 1. Let Assumption 1 be fulfilled. Then, for any $\tau > 0$, there exists a number $h_0 > 0$ such that the system (7) admits a Lyapunov function satisfying for any $h > h_0$ and for an arbitrary admissible switching law the conditions of the Razumikhin theorem.

Next, consider the system (8) with asynchronous switching.

Theorem 2. Let Assumption 2 be fulfilled. Then, for any $\tau > 0$, there exists a number $h_0 > 0$ such that the system (8) admits a Lyapunov function satisfying for any $h > h_0$ and for an arbitrary admissible switching law the conditions of the Razumikhin theorem.

Proof. Let the matrix $\tilde{P}$ possess the properties specified in Assumption 2. Construct a Lyapunov function candidate for (8) in the form

$$V(x,y) = \frac{1}{2}\left(x^T\tilde{P}x + \frac{\varepsilon}{h}y^TAy\right), \tag{9}$$

where ε is a positive parameter. The function (9) is positive definite.

Differentiating $V(x,y)$ along the solutions of (8), we obtain

$$\begin{aligned} \dot{V} &= -\frac{1}{h}x^T(t)\tilde{P}G^{-1}\left(C_{\sigma(t)} + D_{\sigma(t-\tau)}\right)x(t) + \frac{1}{h}x^T(t)\tilde{P}G^{-1}D_{\sigma(t-\tau)}\left(x(t) - x(t-\tau)\right) - \frac{\varepsilon}{h}y^T(t)By(t) \\ &- \frac{\varepsilon}{h}y^T(t)\left(C_{\sigma(t)}x(t) + D_{\sigma(t-\tau)}x(t-\tau) - C_{\sigma(t)}(B+hG)^{-1}Ay(t) - D_{\sigma(t-\tau)}(B+hG)^{-1}Ay(t-\tau)\right) \\ &+ x^T(t)\tilde{P}(B+hG)^{-1}\left(\frac{1}{h}BG^{-1}\left(C_{\sigma(t)}x(t) + D_{\sigma(t-\tau)}x(t-\tau)\right) + C_{\sigma(t)}(B+hG)^{-1}Ay(t) + D_{\sigma(t-\tau)}(B+hG)^{-1}Ay(t-\tau)\right) \\ &\le -\frac{\alpha_1}{h}\|x(t)\|^2 + \frac{\alpha_2}{h}\|x(t)\|\|x(t) - x(t-\tau)\| - \frac{\varepsilon\alpha_3}{h}\|y(t)\|^2 + \frac{\alpha_4\varepsilon}{h}\|y(t)\|\left(\|x(t)\| + \|x(t-\tau)\| + \frac{1}{h}\|y(t)\| + \frac{1}{h}\|y(t-\tau)\|\right) \\ &\quad + \frac{\alpha_5}{h^2}\|x(t)\|\left(\|x(t)\| + \|x(t-\tau)\| + \|y(t)\| + \|y(t-\tau)\|\right), \end{aligned}$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ are positive constants.

Consider a solution $\left(x^T(t),y^T(t)\right)^T$ of (8) satisfying the Razumikhin condition: $V(x(\xi),y(\xi)) \le 2V(x(t),y(t))$ for $\xi \in [t-2\tau,t]$. Then there exist numbers $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\|x(\xi)\| \le \beta_1\left(\|x(t)\| + \sqrt{\frac{\varepsilon}{h}}\|y(t)\|\right), \qquad \|y(\xi)\| \le \beta_2\left(\|y(t)\| + \sqrt{\frac{h}{\varepsilon}}\|x(t)\|\right) \tag{10}$$

for $\xi \in [t-2\tau,t]$.

In addition,

$$|x_i(t) - x_i(t-\tau)| = \tau|\dot{x}_i(t-\zeta_i\tau)| \le \beta_3\tau\left(\frac{1}{h}\|x(t-\zeta_i\tau)\| + \frac{1}{h}\|x(t-\zeta_i\tau-\tau)\| + \frac{1}{h^2}\|y(t-\zeta_i\tau)\| + \frac{1}{h^2}\|y(t-\zeta_i\tau-\tau)\|\right), \qquad i = 1,\ldots,n, \tag{11}$$

where $\zeta_i \in (0,1)$ and $\beta_3 = \text{const} > 0$.

Using (10) and (11), we obtain that the derivative of the Lyapunov function (9) along the considered solution satisfies the estimate

$$\dot{V}(x(t),y(t)) \le -\left(\frac{\alpha_1}{h} - \frac{\alpha_6(\tau+1)}{h^2} - \frac{\alpha_7\tau}{h^{5/2}\sqrt{\varepsilon}} - \frac{\alpha_8}{h^{3/2}\sqrt{\varepsilon}}\right)\|x(t)\|^2 - \left(\frac{\varepsilon\alpha_3}{h} - \frac{\alpha_9}{h^2} - \frac{\alpha_{10}\varepsilon^{3/2}}{h^{3/2}} - \frac{\varepsilon\alpha_{11}}{h^2}\right)\|y(t)\|^2 + \alpha_{12}\left(\frac{(\tau+1)\sqrt{\varepsilon}}{h^{5/2}} + \frac{\sqrt{\varepsilon}}{h^{3/2}} + \frac{\tau}{h^3} + \frac{\varepsilon}{h} + \frac{1}{h^2}\right)\|x(t)\|\|y(t)\|.$$

Here $\alpha_j > 0$, $j = 6,\ldots,12$. Verifying the conditions of the Sylvester criterion, it can be shown that, if ε is sufficiently small and h is sufficiently large, then

$$\dot{V}(x(t),y(t)) \le -\frac{1}{2}\left(\frac{\alpha_1}{h}\|x(t)\|^2 + \frac{\varepsilon\alpha_3}{h}\|y(t)\|^2\right).$$

This implies (see [20, 21]) the asymptotic stability of (8). The proof is completed.

Remark 5. Comparing Theorems 1 and 2, we obtain that, for the system (7), the Razumikhin approach provides us less conservative asymptotic stability conditions than those for the system (8).

5. Application of Lyapunov—Krasovskii Functionals

In this section, to derive conditions ensuring asymptotic stability of (7) and (8) under arbitrary admissible switching laws, we will use the Lyapunov—Krasovskii approach.

Theorem 3. Let Assumption 2 be fulfilled and an arbitrary delay $\tau > 0$ be given. Then one can choose a number $h_0 > 0$ such that, for any $h > h_0$ and for any admissible switching law, there exists a Lyapunov—Krasovskii functional guaranteeing the asymptotic stability of the system (7).

Proof. Let the matrix $\tilde{P}$ possess the properties specified in Assumption 2. Construct a Lyapunov—Krasovskii functional candidate for (7) as follows:

$$V(t,x_t,y_t) = \frac{1}{2}\left(x^T(t)\tilde{P}x(t) + \frac{1}{h^2}y^T(t)Ay(t)\right) - \frac{1}{h}x^T(t)\tilde{P}G^{-1}\int_{t-\tau}^{t} D_{\sigma(u+\tau)}x(u)du + \int_{t-\tau}^{t}\left(\frac{\lambda}{h} + \frac{1}{h^2}(u-t+\tau)\right)\|x(u)\|^2 du + \frac{\eta}{h^2}\int_{t-\tau}^{t}\|y(u)\|^2 du.$$

Here λ,η are positive parameters. Differentiating this functional along the solutions of (7), we obtain

$$\begin{aligned}
\dot{V} = {} & -\frac{1}{h}x^T(t)\tilde{P}G^{-1}\left(C_{\sigma(t)} + D_{\sigma(t+\tau)}\right)x(t) + \left(\frac{\lambda}{h} + \frac{\tau}{h^2}\right)\|x(t)\|^2 - \frac{\lambda}{h}\|x(t-\tau)\|^2 - \frac{1}{h^2}y^T(t)By(t) + \frac{\eta}{h^2}\|y(t)\|^2 - \frac{\eta}{h^2}\|y(t-\tau)\|^2 - \frac{1}{h^2}\int_{t-\tau}^{t}\|x(u)\|^2 du \\
& - \frac{1}{h^2}y^T(t)\left(C_{\sigma(t)}x(t) + D_{\sigma(t)}x(t-\tau) - C_{\sigma(t)}(B+hG)^{-1}Ay(t) - D_{\sigma(t)}(B+hG)^{-1}Ay(t-\tau)\right) \\
& + x^T(t)\tilde{P}(B+hG)^{-1}\left(\frac{1}{h}BG^{-1}\left(C_{\sigma(t)}x(t) + D_{\sigma(t)}x(t-\tau)\right) + C_{\sigma(t)}(B+hG)^{-1}Ay(t) + D_{\sigma(t)}(B+hG)^{-1}Ay(t-\tau)\right) \\
& + \frac{1}{h}\left((B+hG)^{-1}\left(C_{\sigma(t)}x(t) + D_{\sigma(t)}x(t-\tau)\right) - (B+hG)^{-1}\left(C_{\sigma(t)}(B+hG)^{-1}Ay(t) + D_{\sigma(t)}(B+hG)^{-1}Ay(t-\tau)\right)\right)^T \tilde{P}G^{-1}\int_{t-\tau}^{t} D_{\sigma(u+\tau)}x(u)du \\
\le {} & -\left(\frac{\alpha_1}{h} - \frac{\lambda}{h} - \frac{\tau}{h^2}\right)\|x(t)\|^2 - \frac{\lambda}{h}\|x(t-\tau)\|^2 - \left(\frac{\alpha_2}{h^2} - \frac{\eta}{h^2}\right)\|y(t)\|^2 - \frac{\eta}{h^2}\|y(t-\tau)\|^2 - \frac{1}{h^2}\int_{t-\tau}^{t}\|x(u)\|^2 du \\
& + \frac{\alpha_3}{h^2}\|y(t)\|\left(\|x(t)\| + \|x(t-\tau)\| + \frac{1}{h}\|y(t)\| + \frac{1}{h}\|y(t-\tau)\|\right) + \frac{\alpha_4}{h^2}\|x(t)\|\left(\|x(t)\| + \|x(t-\tau)\| + \|y(t)\| + \|y(t-\tau)\|\right) \\
& + \frac{\alpha_5}{h^2}\left(\|x(t)\| + \|x(t-\tau)\| + \frac{1}{h}\|y(t)\| + \frac{1}{h}\|y(t-\tau)\|\right)\int_{t-\tau}^{t}\|x(u)\|du.
\end{aligned}$$

Here $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ are positive constants.

Applying the Silvester criterion once again, it is easy to verify that if the values of λ,η are sufficiently small, then there exists $h_0 > 0$ such that the inequalities

$$\mu_1\|x(t)\|^2 + \frac{\mu_2}{h^2}\|y(t)\|^2 + \frac{\mu_3}{h}\int_{t-\tau}^{t}\|x(u)\|^2 du + \frac{\eta}{h^2}\int_{t-\tau}^{t}\|y(u)\|^2 du \le V(t,x_t,y_t) \le \mu_4\|x(t)\|^2 + \frac{\mu_5}{h^2}\|y(t)\|^2 + \frac{\mu_6}{h}\int_{t-\tau}^{t}\|x(u)\|^2 du + \frac{\eta}{h^2}\int_{t-\tau}^{t}\|y(u)\|^2 du,$$

$$\dot{V} \le -\frac{1}{2}\left(\frac{\alpha_1}{h}\|x(t)\|^2 + \frac{\lambda}{h}\|x(t-\tau)\|^2 + \frac{\alpha_2}{h^2}\|y(t)\|^2 + \frac{\eta}{h^2}\|y(t-\tau)\|^2 + \frac{1}{h^2}\int_{t-\tau}^{t}\|x(u)\|^2 du\right)$$

hold for any $h > h_0$, where μ_i are positive constants, $i = 1,\dots,6$. Hence (see [20, 21]), the system (7) is asymptotically stable. The proof is completed.

Finally in this section, define the conditions of the existence of a Lyapunov—Krasovskii functional in the case of asynchron ous switching.

Theorem 4. Let Assumption 1 be fulfilled and an arbitrary delay $\tau > 0$ be given. Then one can choose a number $h_0 > 0$ such that, for any $h > h_0$ and for any admissible switching law, there exists a Lyapunov—Krasovskii functional guaranteeing the asymptotic stability of the system (8).

Proof. Let the matrix P possess the properties specified in Assumption 1. Construct a Lyapunov—Krasovskii functional candidate for (8) as follows:

$$V(t,x_t,y_t) = \frac{1}{2}\left(x^T(t)Px(t) + \frac{1}{h^2}y^T(t)Ay(t)\right) - \frac{1}{h}x^T(t)PG^{-1}\int_{t-\tau}^{t} D_{\sigma(u)}x(u)du + \int_{t-\tau}^{t}\left(\frac{\lambda}{h} + \frac{1}{h^2}(u-t+\tau)\right)\|x(u)\|^2 du + \frac{\eta}{h^2}\int_{t-\tau}^{t}\|y(u)\|^2 du.$$

Here λ,η are positive parameters. Differentiating this functional along the solutions of (8), we obtain

$$\dot{V} = -\frac{1}{h}x^{T}(t)PG^{-1}\left(C_{\sigma(t)} + D_{\sigma(t)}\right)x(t) + \left(\frac{\lambda}{h} + \frac{\tau}{h^2}\right)\|x(t)\|^2 - \frac{\lambda}{h}\|x(t-\tau)\|^2 - \frac{1}{h^2}y^{T}(t)By(t) + \frac{\eta}{h^2}\|y(t)\|^2 - \frac{\eta}{h^2}\|y(t-\tau)\|^2 - \frac{1}{h^2}\int_{t-\tau}^{t}\|x(u)\|^2 du$$
$$-\frac{1}{h^2}y^{T}(t)\left(C_{\sigma(t)}x(t) + D_{\sigma(t-\tau)}x(t-\tau) - C_{\sigma(t)}(B+hG)^{-1}Ay(t) - D_{\sigma(t-\tau)}(B+hG)^{-1}Ay(t-\tau)\right)$$
$$+x^{T}(t)P(B+hG)^{-1}\left(\frac{1}{h}BG^{-1}\left(C_{\sigma(t)}x(t) + D_{\sigma(t-\tau)}x(t-\tau)\right) + C_{\sigma(t)}(B+hG)^{-1}Ay(t) + D_{\sigma(t-\tau)}(B+hG)^{-1}Ay(t-\tau)\right)$$
$$+\frac{1}{h}\left((B+hG)^{-1}\left(C_{\sigma(t)}x(t) + D_{\sigma(t-\tau)}x(t-\tau)\right) - (B+hG)^{-1}\left(C_{\sigma(t)}(B+hG)^{-1}Ay(t) + D_{\sigma(t-\tau)}(B+hG)^{-1}Ay(t-\tau)\right)\right)^{T}PG^{-1}\int_{t-\tau}^{t}D_{\sigma(u)}x(u)du.$$

The subsequent proof is a similar to that of Theorem 3.

Remark 6. Comparing Theorems 3 and 4, we obtain that, for the system (8), the Lyapunov—Krasovskii approach provides us less conservative asymptotic stability conditions than those for the system (7).

6. Results of a Numerical Simulation

Let the systems (1) and (2) be of the form

$$\ddot{q}(t) + \left(b\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + h\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)\dot{q}(t) + C_{\sigma(t)}q(t) + D_{\sigma(t)}q(t-\tau) = 0\,, \tag{12}$$

$$\ddot{q}(t) + \left(b\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + h\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)\dot{q}(t) + C_{\sigma(t)}q(t) + D_{\sigma(t-\tau)}q(t-\tau) = 0\,, \tag{13}$$

respectively. Here $q(t),\dot{q}(t) \in R^2$, $\sigma : [-\tau,+\infty) \mapsto \{1,2\}$ is an admissible switching law, $C_s, D_s \in R^{2\times 2}$ are constant matrices, $s = 1,2$, b and h are positive parameters, $\tau > 0$ is a delay.

Assume that

$$C_1 = \begin{pmatrix}c_{11} & 0\\ 0 & c_{12}\end{pmatrix}, \qquad C_2 = \begin{pmatrix}c_{21} & 0\\ 0 & c_{22}\end{pmatrix},$$

where $c_{ij} < 0$, $i,j = 1,2$. Then (see [25, 26]), in the case where $D_1 = D_2 = 0$, the subsystems associated with (12) and (13) are unstable. Our objective is stabilization of (12), (13) using the terms with delay.

Let

$$D_1 = \begin{pmatrix}-c_{11} & -\omega_1\\ \omega_1 & -c_{12}\end{pmatrix}, \qquad D_2 = \begin{pmatrix}-c_{21} & -\omega_2\\ \omega_2 & -c_{22}\end{pmatrix}.$$

Here ω_1,ω_2 are positive constants. It is worth noting that systems of the form (12), (13) with given matrices C_s, D_s can be used for modeling vertical gyro with radial correction [2]. Such devices are widely applied in aircrafts, in systems of orbital orientation of artificial satellites, for measuring the roll and pitch angles of ships (see Fig. 1).

Consider the corresponding families of precessional subsystems (4) and (5). It is easy to verify that Assumption 1 is fulfilled. Applying Theorems 1 and 4, we obtain that, for any $\tau > 0$, there exists a number $h_0 > 0$ such that the systems (12), (13) are asymptotically stable for any $h > h_0$ and for an arbitrary switching law.

For simulation, we choose $b = 1$, $h = 6$, $\tau = 3/2$, $\omega_1 = 2$, $\omega_2 = 1$, $c_{11} = -1$, $c_{12} = -3$, $c_{21} = -3$ $c_{22} = -1$. Assume that $\sigma(t) = 1$ for $t \in [-\tau, 2)$, $\sigma(t) = 2$ for $t \in [2,4)$, and $\sigma(t+4) = \sigma(t)$ for $t \geq 0$. In Figs. 2 and 3 the behavior of the solution with initial function $\varphi(\xi) = (1,2)^{T}$ for $\xi \in [-\tau, 0]$ is presented for synchronous and asynchronous cases, respectively. The results obtained confirm the theoretical conclusions.

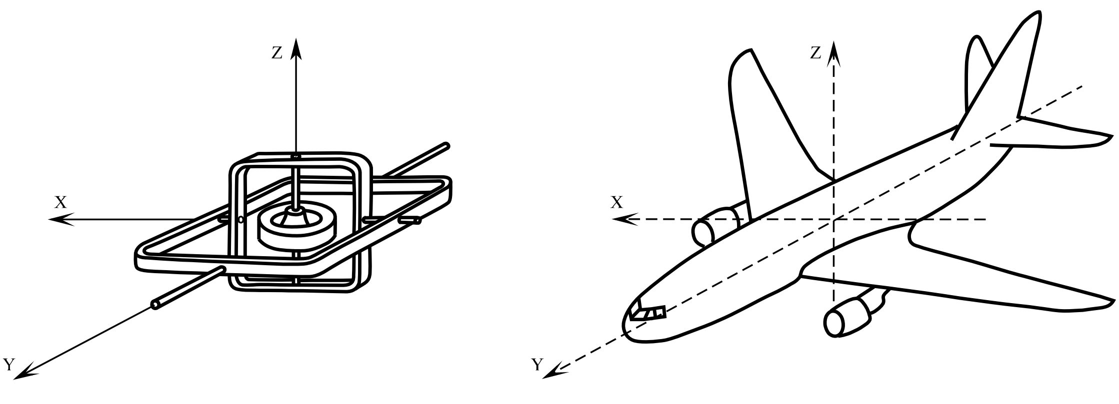

Fig. 1. An aircraft with gyroscopic autopilot

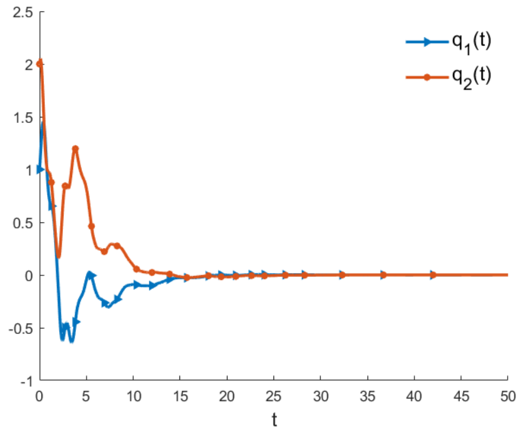

Fig. 2. Synchronous switching

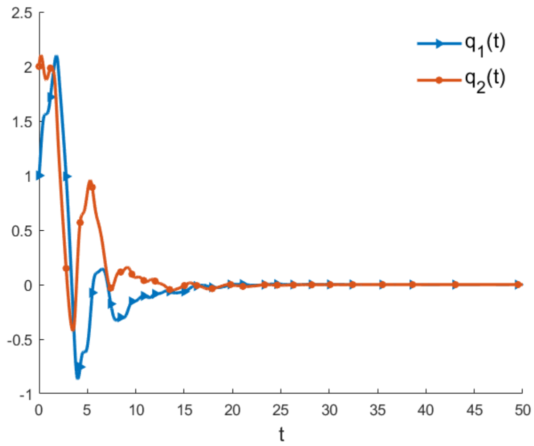

Fig. 3. Asynchronous switching

On the other hand, it should be noted that, for chosen values of parameters, the subsystem

$$h\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\dot{x}(t)+(C_1+D_2)x(t)=0$$

from the corresponding family (5) is unstable. Therefore, Assumption 2 is not fulfilled.

7. Conclusion

The paper was devoted to the stability analysis of linear mechanical systems with dissipative, gyroscopic and switched positional forces. It was assumed that the considered systems contain a constant delay in the positional forces and a large positive parameter as a multiplier at the matrix of gyroscopic forces. Two scenarios were studied: synchronous and asynchronous switching. In both cases, the same stability conditions were obtained. However, it is worth noting that, to derive these conditions, for the system with synchronous switching, the Razumikhin approach should be applied, while, for the system with asynchronous switching, a special construction of Lyapunov—Krasovskii functional should be used. The results of numerical experiments were provided demonstrating the efficiency of the methodology. An important direction for further research is an extension of the developed approaches to gyroscopic systems with time-varying delay.

Author Contributions

A.Yu. Aleksandrov planned the scheme, initiated the project and suggested the experiments; N.A. Stepenko developed the simulations and examined the theory validation. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest concerning the research, authorship, and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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