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Research Paper

Variational Derivation of Truncated Timoshenko-Ehrenfest Beam Theory

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Abstract. The beam theory allowing for rotary inertia and shear deformation and without the fourth order derivative with respect to time as well as without the slope inertia, as was developed by Elishakoff through the dynamic equilibrium consideration, is derived here by means of both direct and variational methods. This formulation is important for using variational methods of Rayleigh, Ritz as well as the finite element method (FEM). Despite the fact that literature abounds with variational formulations of the original Timoshenko-Ehrenfest beam theory, since it was put forward in 1912-1916, until now there was not a single derivation of the version without the fourth derivative and without the slope inertia. This gap is filled by the present paper. It is shown that the differential equations and the corresponding boundary conditions, used to find the solution of the dynamic problem of a truncated Timoshenko-Ehrenfest via variational formulation, have the same form to that obtained via direct method. Finally, in order to illustrate the advantages of the variational approach and its adaptability to the finite element formulation, some numerical examples are performed. The calculations are implemented through a software developed in Mathematica language and results are validated by comparison with those available in the literature.

Keywords: Rotary inertia and shear deformation, variational method, truncated Timoshenko-Ehrenfest model.

1. Introduction

In 1916, Timoshenko [1] published, in his textbook on elasticity, written in the Russian language, a first work on dynamic theory, which takes into account both rotary inertia and shear deformation effect. As Timoshenko stated in the footnote, he developed this theory together with Paul Ehrenfest who lived in St. Petersburg during the years 1907-1912. In 1920 [2], and later in 1921 [3], Timoshenko re-published the derivation in his two papers in English. Timoshenko did not includes Ehrenfest as his co-author due to reasons unknown to present authors. Later on, in 1922 [4], Timoshenko maintained that this theory was developed by Paul Ehrenfest and himself. Detailed history of these developments in the monograph by Elishakoff [5] is given. In [6] Elishakoff showed via derivation of dynamic equilibrium equations, with modification of the original derivation by Timoshenko and Ehrenfest, that the last term in the Timoshenko-Ehrenfest equation, namely the fourth-order derivative was superfluous. Actually, Timoshenko [1-3] found that for simply supported beams, in the characteristic equation, the last term was of the negligible value. Elishakoff [6] demonstrated that the term should be neglected in the differential equation itself. The question arises if the omission of the fourth-order derivative is a simplification or it bears some additional significance. Naturally, it simplifies the derivations greatly. It should be noted that already in 1977 van der Heijden [7] stressed: As T.B.T. [Timoshenko Beam Theory] may be considered as a one-dimensional case of Reissners plate theory, it may yield quite accurate numerical results, even though it is not a consistent theory from the point of view of asymptotic analysis. An asymptotically consistent dynamic equation for a Timoshenko-Ehrenfest-type linear isotropic prismatic beam was apparently first derived by Berdichevsky and Kvashnina [8] using the variational asymptotic approach based on series expansion of the energy functional in 3D elasticity (see also works by Goldenveiser et al. [9] and Berdichevsky [10]). Justification of neglecting the fourth order derivative in Timoshenko-Ehrenfest equations was provided in [12]. In [5] the Timoshenko-Ehrenfest equations without the fourth order time derivative is referred to as the truncated Timoshenko-Ehrenfest equation. The natural question arises about the variational derivation of the Timoshenko-Ehrenfest beam theory. Variational derivation of the original Timoshenko-Ehrenfest equations is given by numerous authors. The interested reader can consult with papers by Carnegie [13], Leech [14], Lee et al. [15], Li and Ho [16], Yu and Hodges [17], Barguev and Mizhidon [18], and possibly others. Dym and Shames [19] and Reddy [20] also provided variational derivation of Timoshenko-Ehrenfest equations in definitive monographs. The papers by Soldatos [21] and Jafarali et al. [22] the equilibrium of dynamic shell theories based upon the variational and vectorial formulation has been presented; papers by Freddi et al. [23], Jemielita [24], Kusuk et al. [25], Shi and Voyiadjis [26], Auciello et al. [27-29], De Rosa, Lippiello and their collaborators [30-34] present much

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interest in this context. In particular, in [30] for the first time De Rosa and Lippiello, using the nonlocal Euler-Bernoulli beam theory, presented a re-formulation of Hamilton's principle and derived the equations of motion and the general corresponding boundary conditions for vibration analysis of nanotube. The novelty of the developed formulation consisted in introducing a potential energy term in the variational approach that considers the work of rotational inertia forces, thus validating the theory formulated by Reddy and Pang [35] obtained geometrically. The pertinent question arises on whether or not the truncated version of Timoshenko-Ehrenfest equation can be obtained variationally. In the paper by Elishakoff et al. [36] there is such an initial attempt taken. They did not obtain the truncated Timoshenko-Ehrenfest version but one additional term that was identified as stemming from slope inertia. This version dealt with in the papers by Bhat et al [37] and Xia et al [38].

The use of Timoshenko-Ehrenfest beam theory, in the field of engineering and nanotechnology was also reported in [39-42] and further applications have been employed in analyzing the buckling and vibration problems in beams, functionally graded beams and nanobeams or nanotubes.

This paper develops the truncated Timoshenko-Ehrenfest model via variational formulation and direct method. According to Tonti [43], "people are divided into two categories: from one side who see the variational formulation of a physical law the apotheosis of economy of nature; from the other side those who deny them any interest and affirm that they can be formulated only after the physical law of the phenomenon is established".

The novelty of the proposed approach is that to show a perfect analogy between variational and direct methods for the dynamic analysis of beams. The aim of the proposed formulations is devoted to finding the truncated Timoshenko-Ehrenfest equations and the corresponding boundary conditions and their mathematical similarity using the two different approaches is established. It is shown that the differential equations and the corresponding boundary conditions, used to find the solution of the dynamic problem of a truncated Timoshenko-Ehrenfest via variational formulation, have the same form to that obtained via direct method. The proposed theory is variationally consistent, so that both the variational approach and the direct geometric approach lead to the same governing equation and to the same boundary conditions. Previously, a similar beam model has been developed by one of the authors, by using the dynamic equilibrium equations of the classical Timoshenko-Ehrenfest beams. Quite arbitrarily, in the resulting differential equation a fourth-order term was considered negligible. Now, this paper gives the exact variational theory for this "truncated beam", and from this point of view closes a gap in the literature. To the authors' knowledge, no analytical formulation have been provided in the literature for this truncated model. Finally, in order to illustrate the advantages of the variational approach and its adaptability to the finite element formulation, some numerical examples are performed. The calculations are implemented and the obtained results are validated by comparison with those available in the literature.

2. Theoretical Formulation

2.1. Variational method for truncated Timoshenko-Ehrenfest beam model

Consider a Timoshenko beam of length L. According to the Hamilton Principle, the motion equations of system are derived as follows:

$$\int_{t_1}^{t_2} \delta\left(V - E_t\right) dt = 0 \tag{1}$$

where δ denotes the variation, t is time, V and E_t denote the kinetic and total potential energy of the beam, respectively.

The kinetic energy of the structure under consideration can be expressed as:

$$V = \frac{1}{2}\int_0^L \rho A \left(\frac{\partial v}{\partial t}\right)^2 dz \tag{2}$$

where $v(z,t)$ is the transverse displacement of the beam, z is the spatial coordinate along the beam, A is the cross-sectional area, ρ the mass density of beam.

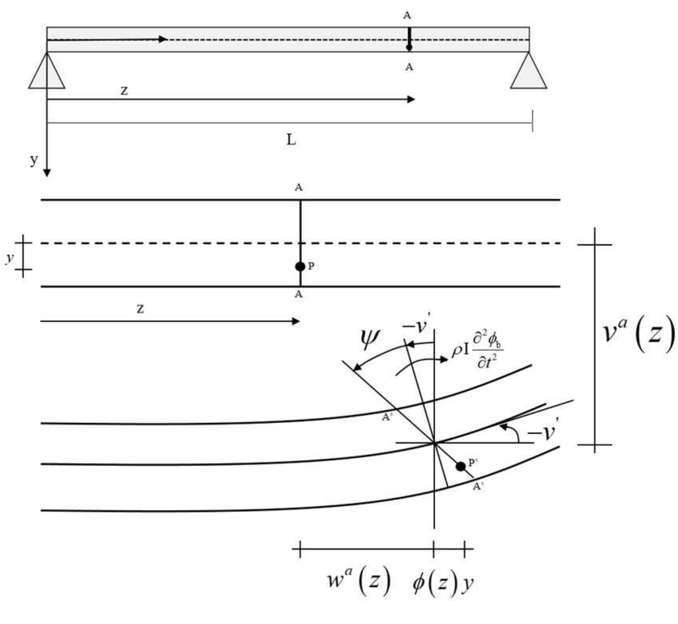

Fig. 1. Geometric representation of rotations variation for Timoshenko beam theory.

The transverse displacement that are acting downwards are considered positive. The strain energy is given by:

$$L_e = \frac{1}{2}\int_0^L EI\left(\frac{\partial \phi}{\partial z}\right)^2 dz + \frac{1}{2}\int_0^L \kappa GA\left(\frac{\partial \upsilon}{\partial z} + \phi\right)^2 dz \tag{3}$$

where ϕ is the slope of the deflection curve due to bending and shear. The rotation acting in counter clock direction is positive. Parameters E, I, G and κ denote the Young modulus, moment of inertia, Lame's second parameter and the shear coefficient, respectively.

Finally, the bending rotation ϕ_b can be expressed as:

$$\phi_b = -\frac{\partial \upsilon}{\partial z} \tag{4}$$

Let us consider the inertial rotational force as a function of the slope of the deflection curve due to bending (see Fig. 1):

$$m_I = -\rho I \frac{\partial^2 \phi_b}{\partial t^2} \tag{5}$$

Let us assume that this force works for the total rotation of Timoshenko beam:

$$\phi = \phi_b + \psi = -\frac{\partial \upsilon}{\partial z} + \psi \tag{6}$$

so that the work reads as:

$$L = -\int_0^L \rho I \frac{\partial^2 \phi_b}{\partial t^2}\phi\, dz = \int_0^L \rho I \frac{\partial^3 \upsilon}{\partial z \partial t^2}\phi\, dz \tag{7}$$

So, the total potential energy is sum of different contributions: the strain energy of the beam (see Eq. (3)) and the potential energy which is equal to the opposite of the work (7):

$$E_t = \frac{1}{2}\int_0^L EI\left(\frac{\partial \phi}{\partial z}\right)^2 dz + \frac{1}{2}\int_0^L \kappa GA\left(\frac{\partial \upsilon}{\partial z} + \phi\right)^2 dz - \int_0^L \rho I \frac{\partial^3 \upsilon}{\partial z \partial t^2}\phi\, dz \tag{8}$$

The first variation of kinetic and total potential energy of the beam can be easily calculated as:

$$\delta V = \int_0^L \rho A \frac{\partial \upsilon}{\partial t}\delta\frac{\partial \upsilon}{\partial t} dz \tag{9}$$

$$\delta E_t = \int_0^L EI\frac{\partial \phi}{\partial z}\delta\frac{\partial \phi}{\partial z}dz + \int_0^L \kappa GA\left(\frac{\partial \upsilon}{\partial z} + \phi\right)\delta\frac{\partial \upsilon}{\partial z}dz + \int_0^L \kappa GA\left(\frac{\partial \upsilon}{\partial z} + \phi\right)\delta\phi\, dz - \int_0^L \rho I \frac{\partial^3 \upsilon}{\partial z \partial t^2}\delta\phi\, dz \tag{10}$$

Hamilton principle becomes:

$$\int_{t_1}^{t_2}\left(\int_0^L \rho A\frac{\partial \upsilon}{\partial t}\delta\frac{\partial \upsilon}{\partial t}dz - \int_0^L EI\frac{\partial \phi}{\partial z}\delta\frac{\partial \phi}{\partial z}dz - \int_0^L \kappa GA\left(\frac{\partial \upsilon}{\partial z}+\phi\right)\delta\frac{\partial \upsilon}{\partial z}dz - \int_0^L \kappa GA\left(\frac{\partial \upsilon}{\partial z}+\phi\right)\delta\phi\, dz + \int_0^L \rho I\frac{\partial^3 \upsilon}{\partial z\partial t^2}\delta\phi\, dz\right)dt = 0 \tag{11}$$

Performing integration by parts and substituting the obtained equations into Eq. (11), one gets:

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^L\left(-\rho A\frac{\partial^2 \upsilon}{\partial t^2} + \kappa GA\left(\frac{\partial^2 \upsilon}{\partial z^2} + \frac{\partial \phi}{\partial z}\right)\right)\delta\upsilon\, dz\, dt + \int_{t_1}^{t_2}\int_0^L\left(\rho I\frac{\partial^3 \upsilon}{\partial z\partial t^2} + EI\frac{\partial^2 \phi}{\partial z^2} - \kappa GA\left(\frac{\partial \upsilon}{\partial z}+\phi\right)\right)\delta\phi\, dz\, dt + \\ &-\int_{t_1}^{t_2}\left[EI\frac{\partial \phi}{\partial z}\delta\phi\right]_0^L + \left[\kappa GA\left(\frac{\partial \upsilon}{\partial z}+\phi\right)\delta\upsilon\right]_0^L dt = 0\end{aligned} \tag{12}$$

Due to the arbitrariness of variations $\delta\upsilon$ and $\delta\phi$, the two following equations of motion can be defined:

$$\begin{aligned}&-\rho A\frac{\partial^2 \upsilon}{\partial t^2} + \kappa GA\left(\frac{\partial^2 \upsilon}{\partial z^2} + \frac{\partial \phi}{\partial z}\right) = 0\\ &\rho I\frac{\partial^3 \upsilon}{\partial z\partial t^2} + EI\frac{\partial^2 \phi}{\partial z^2} - \kappa GA\left(\frac{\partial \upsilon}{\partial z}+\phi\right) = 0\end{aligned} \tag{13}$$

together with the following general boundary conditions:

$$EI\frac{\partial \phi(0,t)}{\partial z}\delta\phi(0,t) = 0 \tag{14}$$

$$EI\frac{\partial \phi(L,t)}{\partial z}\delta\phi(L,t) = 0 \tag{15}$$

$$\kappa GA\left(\frac{\partial \upsilon(0,t)}{\partial z} + \phi(0,t)\right)\delta\upsilon(0,t) = 0 \tag{16}$$

$$\kappa GA\left(\frac{\partial \upsilon(L,t)}{\partial z}+\phi(L,t)\right)\delta\upsilon(L,t)=0 \tag{17}$$

From the first equation of Eqs. (13), the slope deflection function $\partial\phi/\partial z$ is obtained:

$$\frac{\partial\phi}{\partial z}=\frac{\rho A}{\kappa GA}\frac{\partial^2\upsilon}{\partial t^2}-\frac{\partial^2\upsilon}{\partial z^2} \tag{18}$$

and deriving appropriately, one gets:

$$\frac{\partial^3\phi}{\partial z^3}=\frac{\rho A}{\kappa GA}\frac{\partial^4\upsilon}{\partial z^2\partial t^2}-\frac{\partial^4\upsilon}{\partial z^4} \tag{19}$$

Also, deriving the second equation of Eqs. (13), respect to abscissa z, one obtains:

$$\rho I\frac{\partial^4\upsilon}{\partial z^2\partial t^2}+EI\frac{\partial^3\phi}{\partial z^3}-\kappa GA\left(\frac{\partial^2\upsilon}{\partial z^2}+\frac{\partial\phi}{\partial z}\right)=0 \tag{20}$$

and substituting Eqs. (18-19) into Eq. (20), one obtains a single equation in υ:

$$EI\frac{\partial^4\upsilon}{\partial z^4}+\rho A\frac{\partial^2\upsilon}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4\upsilon}{\partial z^2\partial t^2}=0 \tag{21}$$

Equtation (21) has the same form as the Eq. (24) of the paper [12] by Elisakhoff et al..

2.2. Direct method for truncated Timoshenko-Ehrenfest beam model

According to the Timoshenko-Ehrenfest beam theory, the total rotation of the cross section is given by:

$$\phi=\phi_b+\psi=-\frac{\partial\upsilon}{\partial z}+\psi \tag{22}$$

Moreover, the constitutive equations read as:

$$\begin{aligned}&M=EI\frac{\partial\phi}{\partial z}\\&T=\kappa GA\psi=\kappa GA\left(\frac{\partial\upsilon}{\partial z}+\phi\right)\end{aligned} \tag{23}$$

If we consider the beam element in Fig. 2, at the abscissa z, we should impose the equilibrium of the applied loads:
Equilibrium of the vertical forces:

$$\frac{\partial T}{\partial z}-\rho A\frac{\partial^2\upsilon}{\partial t^2}=0 \tag{24}$$

Equilibrium of the couples around the right section:

$$\frac{\partial M}{\partial z}-T+\rho I\frac{\partial^3\upsilon}{\partial z\partial t^2}=0 \tag{25}$$

where the rotational inertia term depends on the ϕ_b rotation, instead of the total rotation ϕ. If the equations (23) are inserted into Eqs. (24) and (25), we obtain:

$$\begin{aligned}&-\rho A\frac{\partial^2\upsilon}{\partial t^2}+\kappa GA\left(\frac{\partial^2\upsilon}{\partial z^2}+\frac{\partial\phi}{\partial z}\right)=0\\&\rho I\frac{\partial^3\upsilon}{\partial z\partial t^2}+EI\frac{\partial^2\phi}{\partial z^2}-\kappa GA\left(\frac{\partial\upsilon}{\partial z}+\phi\right)=0\end{aligned} \tag{26}$$

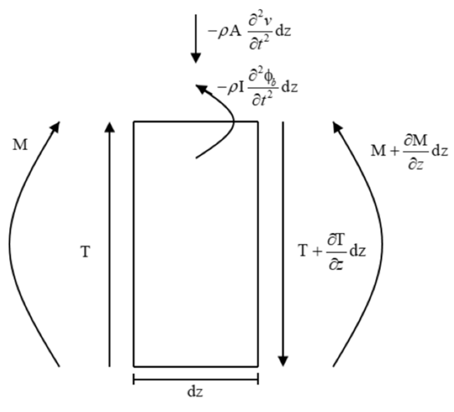

Fig. 2. Beam element.

As can be seen, the Eqs. (26) has the same form as the Eqs. (13) obtained by the variational formulation. It is interesting to note that the governing differential equation for truncated Timoshenko-Ehrenfest beam theory via direct method has already been derived in [38].

3. Discussion

3.1. Variational method for truncated Timoshenko-Ehrenfest beam model

Variational derivation of Timoshenko-Ehrenfest equations is obtained by using the following expression of the kinetic energy:

$$V=\frac{1}{2}\int_0^L \rho A\left(\frac{\partial v}{\partial t}\right)^2 dz+\frac{1}{2}\int_0^L \rho I\left(\frac{\partial \phi}{\partial t}\right)^2 dz \tag{27}$$

The interested readers can consult with texts by Dym and Shames [19], Rao [44], Craig and Kurdila [45], in conjunction with the expression for potential energy given in Eq. (3). In order to derive improved equations of motion, Elishakoff et al. [35] adopted the following expression for the kinetic energy

$$V=\frac{1}{2}\int_0^L \rho A\left(\frac{\partial v}{\partial t}\right)^2 dz+\frac{1}{2}\int_0^L \rho I\left(\frac{\partial^2 v}{\partial t\partial z}\right)^2 dz \tag{28}$$

with the same expression of the potential, strain energy as in Eq. (3). This leads to the following equations (see Eq. (58) in [35]):

$$EI\frac{\partial^4 v}{\partial z^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa G}\right)\frac{\partial^4 v}{\partial z^2\partial t^2}+\frac{\rho EI^2}{\kappa GA}\frac{\partial^6 v}{\partial z^4\partial t^2}=0 \tag{29}$$

We immediately derive that the latter equation contains an additional term, specifically the last one that does not appear in Eq. (28). Authors dubbed Eq. (29) as slope-inertia based version of the Timoshenko-Ehrenfest equation. The slope-inertia based version was studied recently in [46] and [47].The question arose if it was possible to obtain the truncated Timoshenko-Ehrenfest equations without slope-inertia term. Indeed, the derivation of Timoshenko Ehrenfest equations by asymptotic means (see [12]) does not contain the slope inertia term. The present paper provides with the variational formulation that does not lead to slope inertia term but directly results in the truncated version of Timoshenko-Ehrenfest beam, as articulated in [6].

3.2. Direct method for truncated Timoshenko-Ehrenfest beam model

In [6] Elishakoff obtained an equation both more consistent and simpler than the Bresse-Timoshenko equation given by the following expression:

$$EI\frac{\partial^4 v}{\partial z^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I\left(1+\frac{\rho E}{\kappa G}\right)\frac{\partial^4 v}{\partial z^2\partial t^2}+\frac{\rho^2 I}{\kappa G}\frac{\partial^4 v}{\partial t^4}=0 \tag{30}$$

Since the last term of Eq. (30) is obviously of higher order it can be omitted and its elimination leads to the following equation:

$$EI\frac{\partial^4 v}{\partial z^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I\left(1+\frac{\rho E}{\kappa G}\right)\frac{\partial^4 v}{\partial z^2\partial t^2}=0 \tag{31}$$

namely truncated Timoshenko-Ehrenfest equation, whose the corresponding boundary conditions are the same as those with the classical beam theory. The question arose if it was possible to obtain the truncated Timoshenko-Ehrenfest equation by means variational and direct methods.

3.3. Finite element method for truncated Timoshenko-Ehrenfest beam model

In this section, the finite element method for truncated Timoshenko-Ehrenfest beam model is developed, according to the Dawe's formulation [48].

The element model, having 3-nodes and two degrees of freedom per node, is represented by the following interpolated polynomial functions:

$$v(z) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + A_5 z^5 \tag{32}$$

$$\phi(z) = B_0 + B_1 z + B_2 z^2 + B_3 z^3 + B_4 z^4 \tag{33}$$

By using the second equation of (13) and neglecting the rotary inertia term, one gets:

$$EI\frac{\partial^2 \phi}{\partial z^2}=\kappa GA\left(\frac{\partial v}{\partial z}+\phi\right) \tag{34}$$

and the coefficients **g** = $\{B_0,\ldots, B_4\}$ can be expressed in terms of the coefficients **b** = $\{A_0,\ldots, A_5\}$ by equating coefficients of powers of z.

This procedure yields:

$$\begin{aligned}&B_0=-A_1-6\varepsilon A_3-120\varepsilon^2 A_5\\ &B_1=-2(A_2+12\varepsilon A_4);\quad B_2=-3(A_3+20\varepsilon A_5)\\ &B_3=-4A_4;\qquad B_4=-5A_5\end{aligned} \tag{35}$$

where $\varepsilon=EI/\kappa GA$. In matrix form, Eq. (35) becomes:

$$\mathbf{f}=\mathbf{R}\,\mathbf{b} \tag{36}$$

where **f** = $\{A_0,\ldots, A_5, B_0,\ldots, B_4\}$ and **R** is an 11 x 6 rectangular matrix.

Consider the finite element with three nodes whose nodal displacements are given by:

$$\mathbf{d} = \{\upsilon_1, \phi_1, \upsilon_2, \phi_2, \upsilon_3, \phi_3\} \tag{37}$$

and imposing the boundary conditions gets:

$$\mathbf{d} = \mathbf{C}\,\mathbf{b}; \quad \mathbf{b} = \mathbf{C}^{-1}\,\mathbf{d} \tag{38}$$

for z = -L/2, z = 0; z = L/2. For the uniform Timoshenko beam model, the total potential energy, E_t, and the kinetic energy, V, are:

$$\begin{aligned} L_e &= \frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{K}\mathbf{d} \\ \mathrm{D} &= \omega^2\mathbf{d}^{\mathrm{T}}\mathbf{M}_1\mathbf{d} + \frac{1}{2}\omega^2\mathbf{d}^{\mathrm{T}}\mathbf{M}_2\mathbf{d} \end{aligned} \tag{39}$$

where

$$\mathbf{K} = \left(\mathbf{C}^{-1}\right)^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\left(\mathrm{EI}\int_{-L/2}^{L/2}\mathbf{D}_1^{\mathrm{T}}\mathbf{D}_1\,\mathrm{dz} + \kappa\mathrm{GA}\int_{-L/2}^{L/2}\mathbf{D}_2^{\mathrm{T}}\mathbf{D}_3\mathbf{D}_2\,\mathrm{dz}\right)\mathbf{R}\mathbf{C}^{-1} \tag{40}$$

is the stiffness matrix;

$$\mathbf{M}_1 = \left(\mathbf{C}^{-1}\right)^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\left(\rho\mathrm{I}\int_{-L/2}^{L/2}\mathbf{D}_5^{\mathrm{T}}\mathbf{D}_6\,\mathrm{dz}\right)\mathbf{R}\mathbf{C}^{-1} \tag{41}$$

$\mathbf{M}_1$ is potential energy matrix of the rotational inertial loads;

$$\mathbf{M}_2 = \left(\mathbf{C}^{-1}\right)^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\left(\rho\mathrm{A}\int_{-L/2}^{L/2}\mathbf{D}_4^{\mathrm{T}}\mathbf{D}_4\,\mathrm{dz}\right)\mathbf{R}\mathbf{C}^{-1} \tag{42}$$

$\mathbf{M}_2$ is mass matrix related to the kinetic energy;

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 \tag{43}$$

$\mathbf{M}$ is the total mass matrix.

Finally, the matrices $\mathbf{D}_1$ to $\mathbf{D}_5$ are defined as:

$$\begin{aligned}
\mathbf{D}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2z & 3z^2 & 4z^3 \end{pmatrix} \\
\mathbf{D}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & z & z^2 & z^3 & z^4 \\ 0 & 1 & 2z & 3z^2 & 4z^3 & 5z^4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbf{D}_3 &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
\mathbf{D}_4 &= \begin{pmatrix} 1 & z & z^2 & z^3 & z^4 & z^5 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbf{D}_5 &= \begin{pmatrix} 0 & 1 & 2z & 3z^2 & 4z^3 & 5z^4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\mathbf{D}_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & z & z^2 & z^3 & z^4 \end{pmatrix}
\end{aligned} \tag{44}$$

4. Numerical Results

In order to illustrate the advantages of the variational approach and its adaptability to the finite element formulation, some numerical examples are performed. The calculations are implemented through a software developed in Mathematica language [49] and results are validated by comparison with those available in the literature.

In all numerical examples, the structure under consideration is composed of rectangular cross section beam of length L and with the following material properties: mass density ρ = 7860 Kgs^2/m^3, the shear coefficient κ = 0.86667 and Poisson's ratio ν = 0.3, so as taken from [35]. Also, the two parameters characterizing the shear effect are defined as follows:

$$r = \sqrt{\frac{\mathrm{I}}{\mathrm{AL}^2}} = 0.02; \qquad s = \sqrt{\frac{\mathrm{EI}}{\kappa\mathrm{GAL}^2}} = r\sqrt{3} \tag{45}$$

where r is the slenderness ratio.

Finally, following the paper of Elishakoff et al. [35], the first four dimensionless natural frequencies are calculated for different boundary conditions (simply supported, clamped-free and clamped-clamped). The numerical calculations are performed in order to compare the finite element results with those obtained by Elishakoff et al. and reported in [35], for the simply-supported beam case, and with those obtained by the present formulation, for clamped free and clamped-clamped beam cases, respectively.

In the first numerical example, let us consider a simply supported beam and the obtained results are quoted in Table 1.

Table 1. Comparison of the first four dimensionless frequencies from [35] and finite element method, for a simply supported beam

	1 F.E.	2 F.E.	3 F.E.	[35]
Ω_1	9.79662	9.79261	9.79259	9.7925
Ω_2	38.5402	38.3156	38.2888	38.2877
Ω_3	134.5103	83.6666	83.2147	83.1163
Ω_4	298.686	142.709	141.736	141.092

Table 2. Comparison of the first four dimensionless frequencies, as obtained with finite elements, and with the differential equation theory, for a clamped-free beam

	1 F.E.	2 F.E.	3 F.E.	[Truncated]
Ω_1	3.50303	3.50302	3.50302	3.50302
Ω_2	21.6393	21.4838	21.4835	21.4834
Ω_3	60.4262	58.3783	58.2059	58.2021
Ω_4	270.444	111.999	109.378	109.070

Table 3. Comparison of the first four dimensionless frequencies, as obtained with finite elements, and with the differential equation theory, for a clamped-clamped beam

	1 F.E.	2 F.E.	3 F.E.	[Truncated]
Ω_1	21.7797	21.6867	21.6867	21.6867
Ω_2	59.0602	57.6457	57.5254	57.5239
Ω_3	-	109.634	107.77	107.530
Ω_4	-	172.598	169.766	168.275

where $\Omega_i = \sqrt{\rho A \omega_i^2 L^4 / EI}$. As can be seen from Table 1, the obtained numerical and exact results are in an excellent agreement.

In the second numerical example, consider the clamped-free beam. Using the Eq. (21) with the following corresponding boundary conditions, Table 2 lists the first four dimensionless frequencies values and as in the previously reported example, numerical and truncated formulation results are compared, with an excellent agreement found.

Finally, the last numerical example consider the clamped-clamped beam case. The truncated solution, given by the Eq. (21), with those obtained by finite elements procedure, are compared and the variation of the first four dimensionless frequencies values are quoted in Table 3.

The results of Tables 1-3 show high accuracy and efficiency of the proposed element in calculating natural frequencies of beam with different boundary conditions. Also, they confirm that the finite element answers always converge onto the exact results as given by the truncated Timoshenko-Ehrenfest theory. Thus, we conclude that the proposed variational truncated Timoshenko-Ehrenfest model provides a simple and powerful tool in dealing with the vibration analysis of beams.

5. Conclusion

In this study, we derive the truncated Timoshenko-Ehrenfest model via variational formulation and direct method. The advantages of variational setting consists in its adaptability to variational method by Ritz, along with the finite element formulation. Besseling [50] advocates for variational formulation in following words: "*A variational formulation of a theory for physical processes has several advantages . . . the variational formulation provides a sound basis for an approximate formulation of the problem. Theories of beams, plates, and shells, but also finite element models are typical examples of such approximate formulation*". Why is the obtained variational derivation important?

Although there are numerous variational derivations in papers and books of the original Timoshenko-Ehrenfest beam model, there was not a single variational formulation of the truncated version of the Timoshenko-Ehrenfest beam theory. This paper illustrates that Elishakoff's [6] truncated version of the Timoshenko-Ehrenfest beam equations in both asymptotically and variationally valid. Elishakoff, Kaplunov and Nolde demonstrated asymptotic validity of it in [12], whereas variational correctness of it is corroborated by the present study.

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The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Nomenclature

A	The cross-sectional area of rectangular cross-section	I	Moment of inertia
L	Length of beam	ρ	Mass density
E	Young's modulus	κ	Shear coefficient
G	Lame's second parameter	ν	Poisson's ratio

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