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Research Paper

Two Modifications of the Homotopy Perturbation Method for Nonlinear Oscillators

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Abstract: Nonlinear vibration arises in engineering and physics, and the periodic motion of these nonlinear oscillatory systems have rich dynamics. An estimation of amplitude-frequency relationship of a nonlinear oscillator is much needed, therefore, well-known homotopy perturbation method is employed for this purpose. In this paper, two last modifications of the homotopy perturbation method are briefly reviewed, which couples with either the parameter-expansion technology or the enhanced perturbation method. Both modifications are extremely effective for nonlinear oscillators, and the cubic-quintic-septic Duffing oscillator is used as an example to elucidate the solution processes.

Keywords: Homotopy perturbation method, Enhanced perturbation method, Amplitude-frequency relationship, Duffing oscillators, Periodic solution.

1. Introduction

Nonlinear oscillation from the oscillation of a molecule's vibration to the earthquake happens everywhere, and the periodic property performs a critical role in various oscillatory problems of science and enginerring[1-7], for example, the attachment oscillator for the controlled manufacturing of nanofiber membranes[8], Fangzhu's oscillator for collection of water from air[9], harmonic oscillator for micr/nano structures[10], and release oscillator for delivery of ions [11]. In recent years, nonlinear oscillators have received extensive attention from scientific researchers especially after the invention of microelectromechanical systems [12-16]. The exact solutions of these vibratory systems play a crucial role in researching the properties and behavior of periodic motion of the aforesaid systems but identifying such exact solutions is very difficult. Although numerical solutions of such oscillatory systems with nonlinearities are easy to find, researchers are interested in finding the exact or at least analytic solutions to these problems because they have more detail which helps better insight into these systems. Therefore, amplitude-frequency formulation is an essential property of a nonlinear oscillator. There are many analytical techniques used in open literature for amplitude-frequency formulation, for example, the homotopy perturbation method [17-22], the variational iteration method [23-28], the Hamiltonian approach [29-30], He's frequency formulation [31-32] and the max-min approach [33].

The homotopy perturbation method was first introduced in 1998 [17], and it has been widely applied to solve various non-linear problems accurately and efficiently [19-22, 34-38]. The method became famous after 2006 when the review article "some asymptotic methods for strongly nonlinear equations" was published in International Journal of Modern Physics B, which has been cited 1,345 times according to Clarivate's Web of Science (December 7, 2018), the review article gave a lucid as well as elementary introduction to the homotopy perturbation method [21]. On February 2008, Prof. Ganji told Science Watch (http://archive.sciencewatch.com/dr/fbp/2008/08febfbp/08febGanji/) that the homotopy perturbation method is a universal mathematical tool for solving nonlinear problems, and it will become as popular as Newton's iteration method in future. After 2006, publications increased exponentially and now there are about 200 publications per year, see Fig.1. Fractional differential equations [39-41] can be effectively solved by the homotopy perturbation method coupled with Laplace transform, this modification is called as He-Laplace method [42,43].

Though the homotopy perturbation method has been widely adopted, there are still space for further improvement. In this paper we briefly review two modifications using either the parameter expansion technology [44] or the enhanced perturbation method [45-46].

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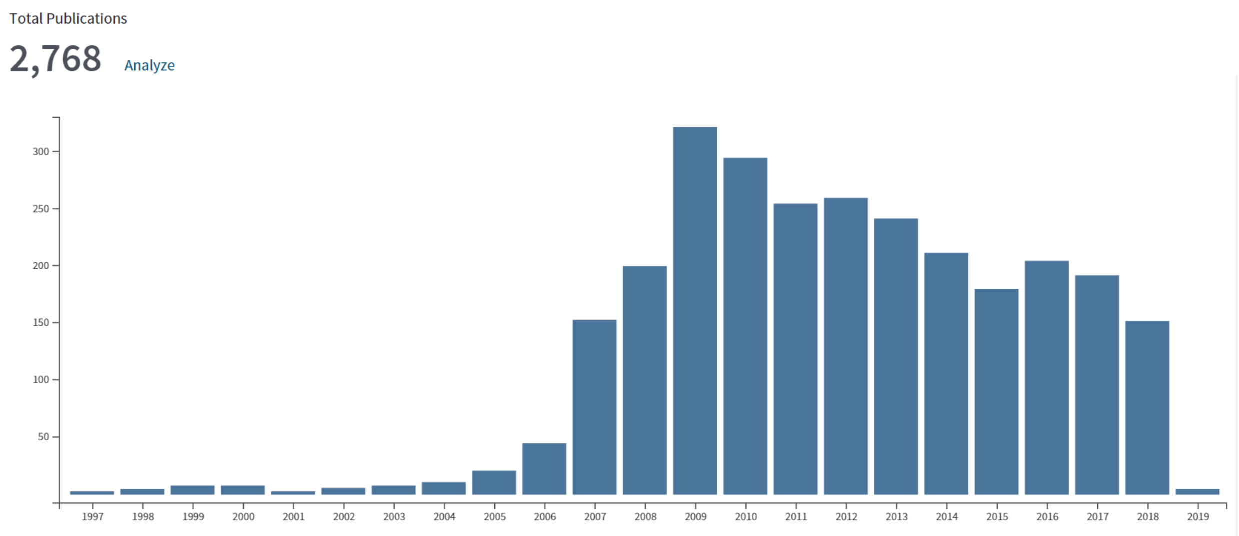

Fig. 1. Publication numbers on "homotopy perturbation method" per year according to Clarivate's Web of Science (December 7, 2018)

2. The homotopy perturbation method with the parameter-expanding method

Consider the general non-linear oscillator equation as

$$x'' + ax + f(x',x) = 0 \tag{1}$$

where $f(x',x)$ is supposed to be a function with non-linear terms, a is a real number, it can be zero or negative. We can construct the homotopy equation for Eq.(1) as

$$x'' + ax + p\left[f(x',x)\right] = 0 \tag{2}$$

The solution can be written as

$$x = x_0 + px_1 + p^2 x_2 + \cdots\cdots \tag{3}$$

According to the parameter-expanding method [20,44] we can expand the coefficient of the linear term in the form

$$a = \Omega^2 + a_1 p + a_2 p^2 + \cdots\cdots \tag{4}$$

where constants Ω^2 and a_i can be identified later.

3. Homotopy perturbation method coupled with the enhanced perturbation method

The perturbation method has been one of the powerful methods used to solve nonlinear problems. Roughly all perturbation theories build on the hypothesis of small parameters. Recently, Filobello-Nino et al. [45] suggested a new perturbation approach to solve the nonlinear equations in order to get the solution not influenced by parameters. Filobello-Nino et al. [45] proposed an adjustment in the perturbation method which was named as the enhanced perturbation method. This method is highly accurate and provides better results, and it can deal problems with both small and large values of the perturbation parameters [46]. To understand the idea of the enhanced perturbation method, we consider the following cubic quintic Duffing oscillator [47]

$$x'' + x + \varepsilon_1 x^3 + \varepsilon_2 x^5 = \cos\beta t \tag{5}$$

Eq. (7) can be expressed as

$$\left(D^2 + 1 + \varepsilon_1 x^2 + \varepsilon_2 x^4\right)x = \cos\beta t \tag{6}$$

where $D = d/dt$ is differential operator.

The enhanced perturbation method [31-32] is to apply the annihilator operator $D^2 + \beta^2$ to Eq. (6), we have

$$\left(D^2 + \beta^2\right)\left(D^2 + 1 + \varepsilon_1 x^2 + \varepsilon_2 x^4\right)x = 0 \tag{7}$$

This method can solve wide class of non-linear problems. It is more effective in case of non-linear problems with forced term but can also apply on problems without forced term [48].

4. Solution of cubic-quintic-septic duffing oscillator

The Duffing equation [47,49-50] is extensively studied non-linear differential equation and related to many engineering systems. In this article, this paper focuses on the cubic-quintic-septic Duffing equation [50]:

$$x'' + \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 = 0 \tag{8}$$

with initial conditions

$$x(0) = A, \qquad x'(0) = 0 \tag{9}$$

where α is coefficient of linear term and β, γ, δ are the cubic, quintic, septic power coefficients respectively. Eq. (8) expresses the Quasi-zero stiffness phenomena which is useful for desiging the applicable structural parameters. This equation looks simple but difficult to solve analytically. The difficulty in obtaining accurate result of Eq. (8) is due to the presence of quintic and septic power non-linearities which make it more complex and complicated. Recently Remmi and Latha [50] obtained an approximate solution using the harmonic balancing method.

We can construct a homotopy equation for Eq. (8) as [20,44]

$$x'' + \alpha x + p\left[\beta x^3 + \gamma x^5 + \delta x^7\right] = 0 \tag{10}$$

The coefficient of linear term (α) and the solution can be expanded into following forms

$$\alpha = \Omega^2 + a_1 p + a_2 p^2 + \cdots\cdots \tag{11}$$

$$x = x_0 + p x_1 + p^2 x_2 + \cdots\cdots \tag{12}$$

Placing Eq. (11) and Eq.(12) into Eq. (10) and proceeding as that by the perturbation method, we have

$$x_0'' + \Omega^2 x_0 = 0 \qquad\qquad x_0(0) = A, \qquad x_0'(0) = 0 \tag{13}$$

$$x_1'' + \Omega^2 x_1 + x_0 a_1 + \beta x_0^3 + \gamma x_0^5 + \delta x_0^7 = 0\,, \quad x_1(0) = 0, \qquad x_1'(0) = 0 \tag{14}$$

Solving x_0 from Eq. (13), we have

$$x_0 = A\cos\Omega t \tag{15}$$

Substituting Eq. (15) into Eq. (14), we obtain

$$x_1'' + \Omega^2 x_1 + a_1 A\cos\Omega t + \beta A^3\cos^3\Omega t + \gamma A^5\cos^5\Omega t + \delta A^7\cos^7\Omega t = 0$$

$$\begin{aligned} x_1'' + \Omega^2 x_1 &+ \left(a_1 A + \frac{3}{4}\beta A^3 + \frac{5}{8}\gamma A^5 + \frac{35}{64}\delta A^7\right)\cos\Omega t + \left(\frac{1}{4}\beta A^3 + \frac{5}{16}\gamma A^5 + \frac{21}{64}\delta A^7\right)\cos 3\Omega t \\ &+ \left(\frac{1}{16}\gamma A^5 + \frac{7}{64}\delta A^7\right)\cos 5\Omega t + \left(\frac{1}{64}\alpha\delta A^7\right)\cos 7\Omega t = 0 \end{aligned} \tag{16}$$

No secular term in x_1 requires

$$a_1 A + \frac{3}{4}\beta A^3 + \frac{5}{8}\gamma A^5 + \frac{35}{64}\delta A^7 = 0 \tag{17}$$

If first order approximation is needed from Eq. (11) we have

$$a_1 = \alpha - \Omega^2 \tag{18}$$

Thus from Eq. (17) and Eq. (18), we have

$$\Omega = \sqrt{\alpha + \frac{3}{4}\beta A^2 + \frac{5}{8}\gamma A^4 + \frac{35}{64}\delta A^6} \tag{19}$$

In order to introduce the enhanced perturbation method [45-46] into the homotopy perturbation method, we first consider the following linear oscillator

$$x'' + \Omega^2 x = (D^2 + \Omega^2)x = 0 \tag{20}$$

According to the enhanced perturbation method [45-46], Eq. (20) can lead to a higher order form:

$$(D^2 + \Omega^2)(D^2 + \Omega^2)x = x'''' + 2\Omega^2 x'' + \Omega^4 = 0 \tag{21}$$

Replacing x'' by $-\alpha\Omega^2$ from Eq. (20) results in

$$x'''' - \Omega^4 x = 0 \tag{22}$$

We adopt the similar way to increase the order of cubic-quintic-septic Duffing equation. To achieve this, differentiating Eq. (8) twice

$$x''' + \alpha x' + 3\beta x^2 x' + 5\gamma x^4 x' + 7\delta x^6 x' = 0 \tag{23}$$

$$x'''' + \alpha x'' + 3\beta(x^2 x'' + 2x x'^2) + 5\gamma(x^4 x'' + 4x^3 x'^2) + 7\delta(x^6 x'' + 6x^5 x'^2) = 0 \tag{24}$$

Replacing x'' by $-(\alpha x + \beta x^3 + \gamma x^5 + \delta x^7)$ from Eq. (8), we have

$$x'''' - \alpha^2 x - \alpha\beta x^3 - \alpha\gamma x^5 - \alpha\delta x^7 + 3\beta(x^2x'' + 2xx'^2) + 5\gamma(x^4x'' + 4x^3x'^2) + 7\delta(x^6x'' + 6x^5x'^2) = 0 \tag{25}$$

or

$$x'''' - \alpha^2 x - \alpha\beta x^3 - \alpha\gamma x^5 - \alpha\delta x^7 + 3\beta x^2x'' + 6\beta xx'^2 + 5\gamma x^4x'' + 20\gamma x^3x'^2 + 7\delta x^6x'' + 42\delta x^5x'^2 = 0 \tag{26}$$

A homotopy equation can be established for Eq. (26) as

$$x'''' - \alpha^2 x + p\left[-\alpha\beta x^3 - \alpha\gamma x^5 - \alpha\delta x^7 + 3\beta x^2x'' + 6\beta xx'^2 + 5\gamma x^4x'' + 20\gamma x^3x'^2 + 7\delta x^6x'' + 42\delta x^5x'^2\right] = 0 \tag{27}$$

The solution can be expanded as Eq. (12) and the coefficient of linear term can be expanded as

$$\alpha^2 = \Omega^4 + p\Omega_1 + p^2\Omega_2 + \cdots\cdots \tag{28}$$

Substituting Eq. (12) and Eq. (28) into Eq. (27) and continuing as that by the perturbation method, we have

$$x_0'''' - \Omega^4 x_0 = 0\,, \qquad x_0(0) = A, \qquad x_0'(0) = 0 \tag{29}$$

$$x_1'''' - \Omega^4 x_1 - \Omega_1 x_0 - \alpha\beta x_0^3 - \alpha\gamma x_0^5 - \alpha\delta x_0^7 + 3\beta x_0^2x_0'' + 6\beta x_0x_0'^2 + 5\gamma x_0^4x_0'' + 20\gamma x_0^3x_0'^2 + 7\delta x_0^6x_0'' + 42\delta x_0^5x_0'^2 = 0, \qquad x_1(0) = 0, \qquad x_1'(0) = 0 \tag{30}$$

Using Eq. (15) as its initial approximate solution, we have

$$\begin{gathered} x_1'''' - \Omega^4 x_1 - A\Omega_1\cos\Omega t - \alpha\beta A^3\cos^3\Omega t - \alpha\gamma A^5\cos^5\Omega t - \alpha\delta A^7\cos^7\Omega t - 3A^3\Omega^2\beta\cos^3\Omega t + 6A^3\Omega^2\beta\cos\Omega t\sin^2\Omega t - 5A^5\Omega^2\gamma\cos^5\Omega t \\ +20A^5\Omega^2\gamma\cos^3\Omega t\sin^2\Omega t - 7A^7\Omega^2\delta\cos^7\Omega t + 42A^7\Omega^2\delta\cos^5\Omega t\sin^2\Omega t = 0 \end{gathered} \tag{31}$$

By means of simple calculation, we conclude

$$\begin{gathered} x_1'''' - \Omega^4 x_1 + (6A^3\Omega^2\beta - A\Omega_1)\cos\Omega t + (20A^5\Omega^2\gamma - 9A^3\Omega^2\beta - \alpha\beta A^3)\cos^3\Omega t \\ +(42A^7\Omega^2\delta - 25A^5\Omega^2\gamma - \alpha\gamma A^5)\cos^5\Omega t - (49A^7\Omega^2\delta + \alpha\delta A^7)\cos^7\Omega t = 0 \end{gathered} \tag{32}$$

or after applying trigonometry formulas, we have

$$\begin{gathered} x_1'''' - \Omega^4 x_1 + \left(-\frac{1}{64}\alpha\delta A^7 - \frac{49}{64}\delta A^7\Omega^2\right)\cos 7\Omega t \\ +\left(-\frac{1}{16}\alpha\gamma A^5 - \frac{7}{64}\alpha\delta A^7 - \frac{25}{16}\gamma A^5\Omega^2 - \frac{175}{64}\delta A^7\Omega^2\right)\cos 5\Omega t + \\ \left(-\frac{1}{4}\alpha\beta A^3 - \frac{5}{16}\alpha\gamma A^5 - \frac{21}{64}\alpha\delta A^7 - \frac{9}{4}\beta A^3\Omega^2 - \frac{45}{16}\gamma A^5\Omega^2 - \frac{189}{64}\delta A^7\Omega^2\right)\cos 3\Omega t \\ +\left(-A\Omega_1 - \frac{3}{4}\alpha\beta A^3 - \frac{5}{8}\alpha\gamma A^5 - \frac{35}{64}\alpha\delta A^7 - \frac{3}{4}\beta A^3\Omega^2 - \frac{5}{8}\gamma A^5\Omega^2 - \frac{35}{64}\delta A^7\Omega^2\right)\cos\Omega t = 0 \end{gathered} \tag{33}$$

Requirement of no secular term needs

$$-A\Omega_1 - \frac{3}{4}\alpha\beta A^3 - \frac{5}{8}\alpha\gamma A^5 - \frac{35}{64}\alpha\delta A^7 - \frac{3}{4}\beta A^3\Omega^2 - \frac{5}{8}\gamma A^5\Omega^2 - \frac{35}{64}\delta A^7\Omega^2 = 0 \tag{34}$$

If the first order approximation is enough, then from Eq. (28), we have

$$\Omega_1 = \alpha^2 - \Omega^4 \tag{35}$$

Solving Ω from Eqs. (34) and (35), we obtain

$$\Omega = \sqrt{\alpha + \frac{3}{4}\beta A^2 + \frac{5}{8}\gamma A^4 + \frac{35}{64}\delta A^6} \tag{36}$$

which is same as that obtained by the homotopy perturbation method with the parameter- expanding method.
Thus our approximate solution in both cases is

$$x(t) = A\cos\left(\sqrt{\alpha + \frac{3}{4}\beta A^2 + \frac{5}{8}\gamma A^4 + \frac{35}{64}\delta A^6}\; t\right) \tag{37}$$

Figure 2 shows the comparison of results obtained from both above explained methods with the numerical one with the help of MATLAB for the cubic-quintic-septic Duffing oscillator for some given parameters.We depict the approximate solution obtained from modifications of homotopy perturbation method (red circles) Eq. (37) with time for four different amplitudes $A = 0.05, 0.1, 0.15, 0.2$ against the parameter values $\alpha = \beta = \gamma = \delta = 1$ with the same yielded by computationally using MATLAB (solid black lines). All panels are showing good accuracy which validate the approximate solution obtained by homotopy perturbation method.

5. Conclusion

In this article, we successfully applied the two modifications of the homotopy perturbation method to the cubic-quintic-septic Duffing oscillator, both modifications result in a same result, giving a cross-proof of the correctness of the both modifications. Comparison with the numerical result show both modifications lead to a high accuracy of the obtained solution.

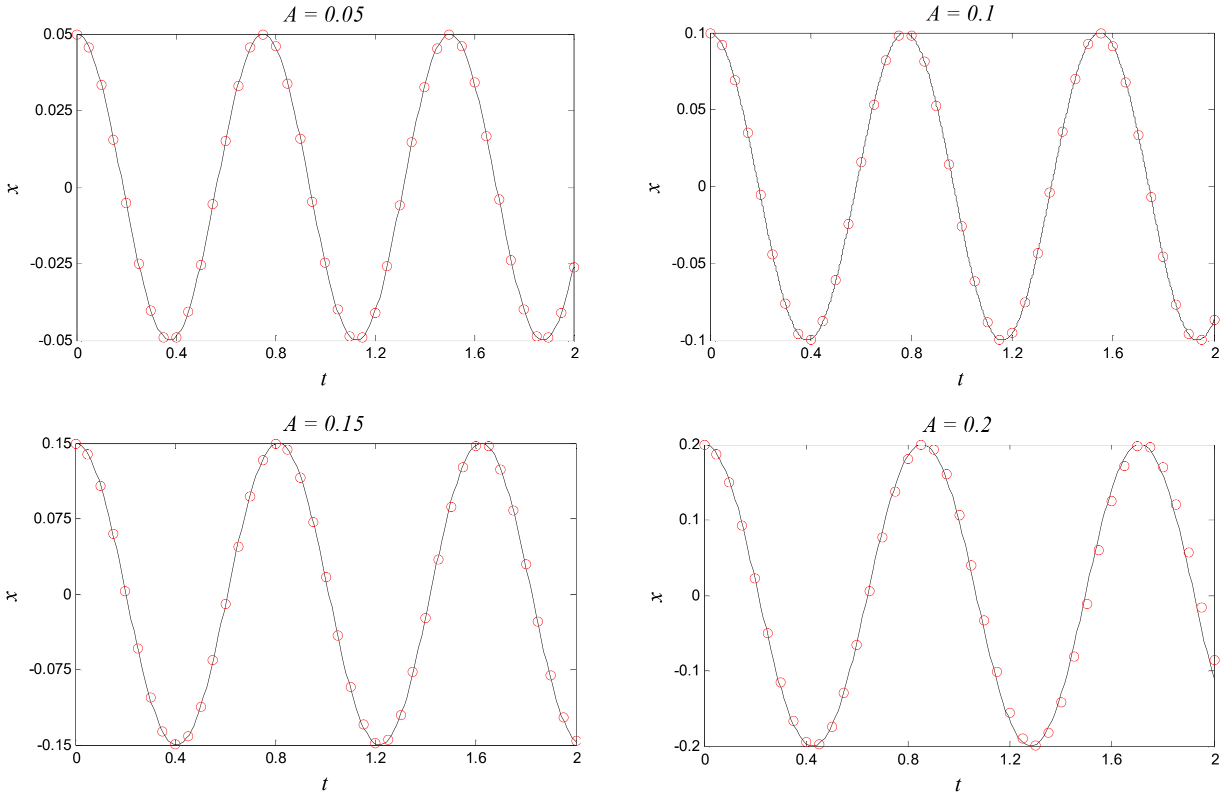

Fig. 2. Comparison of approximate solution with MATLAB solution for the cubic-quintic-septic oscillator

Author Contributions

All authors contribute equally in preparation of this manuscript. All authors discussed the results, reviewed and approved the final version of the manuscript.

Conflict of Interest

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Nomenclature

A	Amplitide of the oscillator	β	Cubic power coefficient
p	Expansion parameter	γ	Quintic power coefficient
x	Approximate solution	δ	Septic power coefficient
t	Time	Ω	Nonlinear frequency of the oscillator
α	Coefficient of linear term		

References

[1] Torrejon, J., Riou, M., Araujo, F. et al., Neuromorphic computing with nanoscale spintronic oscillators, *Nature*, 547, 2017, 428–431.
[2] Sedighi, H.M., Reza, A., High precise analysis of lateral vibration of quintic nonlinear beam, *Latin American Journal of Solids and Structures*, 10(2), 2013, 441-452.
[3] Sedighi, H.M., Shirazi, K.H., Noghrehabadi, A., Application of Recent Powerful Analytical Approaches on the Non-Linear Vibration of Cantilever Beams, *International Journal of Nonlinear Sciences and Numerical Simulation*, 13(7-8), 2012, 487–494.
[4] Romera, M., Talatchian, P., Tsunegi, S. et al., Vowel recognition with four coupled spin-torque nano-oscillators, *Nature*, 563, 2018, 230–234.
[5] Zhao, J.H., Li, X.X., Liu, Z. Needle's vibration in needle-disk electrospinning process: Theoretical model and experimental verification, *Journal of Low Frequency Noise Vibration and Active Control*, 38(3-4), 2019, 1338-1344.
[6] Sedighi, H.M., Reza, A., Zare, J., The effect of quintic nonlinearity on the investigation of transversely vibrating buckled Euler-Bernoulli beams, *Journal of Theoretical and Applied Mechanics*, 51(4), 2013, 959-968.
[7] Gupta, S., Pramanik, S., Smita, Pramanik, A., Analytic approach for love wave dispersion in an inhomogeneous layer lying over an irregular porous half-space, *TWMS Journal of Applied and Engineering Mathematics*, 9(3), 2019, 626-637.
[8] Li, X.X., He, J.H., Nanoscale adhesion and attachment oscillation under the geometric potential. Part 1: The formation mechanism of nanofiber membrane in the electrospinning, *Results in Physics*, 12, 2019, 1405-1410.
[9] He, C.H., He, J.H., Sedighi, H.M., Fangzhu (方诸): An ancient Chinese nanotechnology for water collection from air: History, mathematical insight, promises, and challenges, *Mathematical Methods in the Applied Sciences*, 2020, https://doi.org/10.1002/mma.6384.
[10] Jin, X., Liu, M., Pan, F., Li, Y., Fan, J., Low frequency of a deforming capillary vibration, part 1: Mathematical model, *Journal of Low Frequency Noise Vibration and Active Control*, 38(3-4), 2019, 1676-1680.
[11] Liu, H.Y., Li, Z..M., Yao, Y.J., A fractional nonlinear system for release oscillation of silver ions from hollow fibers, *Journal of Low Frequency Noise*

Vibration and Active Control, 38(1), 2019, 88-92.
[12] Barnard, A.W., Zhang, M., Wiederhecker, G.S. et al., Real-time vibrations of a carbon nanotubes, *Nature*, 566, 2019, 89–93.
[13] Sedighi, H.M., Modeling of surface stress effects on the dynamic behavior of actuated non-classical nano-bridges, *Transactions of the Canadian Society for Mechanical Engineering*, 39, 2015, 137-151.
[14] He, J.H., Skrzypacz, P., Wei, D.M., Dynamic pull-in for micro-electromechanical device with a current-carrying conductor, *Journal of Low Frequency Noise Vibration and Active Control*, 2019, DOI: 10.1177/1461348419847298.
[15] Anjum, N., He, J.H., Nonlinear dynamic analysis of vibratory behavior of a graphene nano/microelectromechanical system, *Mathematical Methods in the Applied Sciences*, 2020, DOI: 10.1002/mma.6699.
[16] Anjum. N., He, J.H., Analysis of nonlinear vibration of nano/microelectromechanical system switch induced by electromagnetic force under zero initial conditions, *Alexandria Engineering Journal*, 2020, https://doi.org/10.1016/j.aej.2020.07.039.
[17] He, J.H., Homotopy perturbation technique, *Computer Methods in Applied Mechanics and Engineering*, 178, 1999, 257-262.
[18] He, J.H., Homotopy perturbation method with two expanding parameters, *Indian Journal of Physics*, 88, 2014, 193–196.
[19] Anjum. N., He, J.H., Homotopy perturbation method for N/MEMS oscillators, *Mathematical Methods in the Applied Sciences*, 2020, DOI: 10.1002/mma.6583.
[20] He, J.H., Some asymptotic methods for strongly nonlinear equations, *International Journal of Modern Physics B*, 20, 2006, 1141-1199.
[21] He, J.H., Homotopy perturbation method for solving boundary value problems, *Physics Letters A*, 350, 2006, 87-88.
[22] He, J.H., New interpretation of homotopy perturbation method, *International Journal of Modern Physics*, 20, 2006, 2561–2568.
[23] He, J.H., Variational iteration method – a kind of non-linear analytical technique: some examples, *International Journal of Nonlinear Mechanics*, 34, 1999, 699–708.
[24] He, J.H., Variational iteration method – some recent results and new interpretations, *Journal of Computational and Applied Mathematics*, 207, 2007, 3–17.
[25] Tao, Z.L., Chen, G.H., Chen, Y.H., Variational iteration method with matrix Lagrange multiplier for nonlinear oscillators, *Journal of Low Frequency Noise Vibration and Active Control*, 2019, DOI:10.1177/1461348418817868.
[26] Anjum, N. He, J.H., Laplace transform: making the variational iteration method easier, *Applied Mathematics Letters*, 2019, 92, 134–138.
[27] Ahmad, H., Variational iteration algorithm-I with an auxiliary parameter for wave-like vibration equations, *Journal of Low Frequency Noise Vibration and Active Control*, 2019, DOI: 10.1177/1461348418823126.
[28] Anjum, N., et al., Numerical iteration for nonlinear oscillators by Elzaki transform, *Journal of Low Frequency Noise Vibration and Active Control*, 2019, https://doi.org/10.1177/1461348419873470.
[29] He, J.H., Hamiltonian approach to nonlinear oscillators, *Physics Letters A*, 374, 2010, 2312–2314.
[30] Zhang, J., Locally exact discretization method for nonlinear oscillation systems, *Journal of Low Frequency Noise Vibration and Active Control*, 2019, DOI: 10.1177/1461348418817125
[31] He, J.H., An improved amplitude-frequency formulation for nonlinear oscillators, *International Journal of Nonlinear Science and Numerical Simulation*, 9, 2008, 211–212.
[32] He, J.H., Amplitude–frequency relationship for conservative nonlinear oscillators with odd nonlinearities, *International Journal of Applied and Computational Mathematics*, 3, 2017, 1557–1560.
[33] Liu, Y.Q., He, J.H., On relationship between two ancient Chinese algorithms and their application to flash evaporation, *Results in Physics*, 7, 2017, 320–322.
[34] Anjum N., Ain Q.T., Application of He's fractional derivative and fractional complex transform for time fractional Camassa-Holm equation, *Thermal Science*, 2019, https://doi.org/10.2298/TSCI190930450A.
[35] Wang, Q.L., et. al.. Fractal calculus and its application to explanation of biomechanism of polar bear hairs, *Fractals*, 26, 2018, 1850086.
[36] Ain, Q.T., et al., The Fractional complex transform: A novel approach to the time-fractional Schrodinger equation, *Fractals*, 2020, https://doi.org/10.1142/S0218348X20501418
[37] Wu, Y., He, J.H., Homotopy perturbation method for nonlinear oscillators with coordinate dependent mass, *Results in Physics*, 10, 2018, 270–271.
[38] Shqair, M., Solution of different geometries reflected reactors neutron diffusion equation using the homotopy perturbation method, *Results in Physics*, 12, 2019, 61–66.
[39] Wei, C., Solving time-space fractional Fitzhugh-Nagumo equation by using He-Laplace decomposition method, *Thermal Science*, 22, 2018, 1723-1728.
[40] Liu, Z.J., et. al. Hybridization of homotopy perturbation method and Laplace transformation for the partial differential equations, *Thermal Science*, 21, 2017, 1843–1846.
[41] Suleman, M., et al., He–Laplace method for general nonlinear periodic solitary solution of vibration equations, *Journal of Low Frequency Noise Vibration and Active Control*, 38(3-4), 2018, 1297-1304.
[42] Mishra, H.K., Nagar, A., He-Laplace Method for Linear and Nonlinear Partial Differential Equations, *Journal of Applied Mathematics*, 2012, 2012, 180315.
[43] Filobello-Nino, U., Vazquez-Leal, H., Rashidi, M.M., et al., Laplace transform homotopy perturbation method for the approximation of variational problems, *SpringerPlus*, 5, 2016, 276.
[44] Shou, D.H., He, J.H., Application of parameter-expanding method to strongly nonlinear oscillators, *International Journal of Nonlinear Science and Numerical Simulation*, 8, 2007, 121–124.
[45] Filobello-Nino, U., et. al., Enhanced classical perturbation method, *Nonlinear Science Letters A*, 9, 2018, 172–185.
[46] Li, X.X., He, C.H., Homotopy perturbation method coupled with the enhanced perturbation method, *Journal of Low Frequency Noise Vibration and Active Control*, 38(3-4), 2019, 1399-1403.
[47] Razzaq, M.A., An analytical approximate technique for solving cubic–quintic Duffing oscillator, *Alexandria Engineering Journal*, 55, 2016, 2959–2965.
[48] Adamu, M.Y., Ogenyi, P., Parameterized homotopy perturbation method, *Nonlinear Science Letters A*, 8, 2017, 240–243.
[49] El-Dib, Y., Stability Analysis of a Strongly Displacement Time-Delayed Duffing Oscillator Using Multiple Scales Homotopy Perturbation Method, *Journal of Applied and Computational Mechanics*, 4(4), 2018, 260-274.
[50] Remmi, S.K., Latha, M.M., Cubic quintic septic duffing oscillator: An analytic study, *Chinese Journal of Physics*, 56, 2018, 2085-2094.

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