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Nonlinear dynamics and stability of a homogeneous model of tall buildings under resonant action

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Abstract. A homogeneous model of beam-like structure, roughly portraying the mechanical behavior of a tall building, is considered to address nonlinear dynamic response in case of external resonant excitation. A symmetric layout of the building is considered, so as to allow the existence of an in-plane response, whose features are evaluated by means of the Multiple Scale Method and accounting for internal resonance, necessarily occurring in the model. Furthermore, to take into account the three-dimensional nature of the problem, stability of the in-plane response to out-of-plane disturbances is addressed, solving the associated parametrically excited linear system.

Keywords: Homogeneous model, tall building, nonlinear dynamics, stability, perturbation methods.

1. Introduction

Homogeneous formulation can often represent a convenient alternative to detailed, refined and costly models, when the target is to roughly describe the mechanical features of very complicated lattice bodies. In this framework, equivalent continua are suitably used to model periodic [1, 2] and pantographic structures [3, 4]. In specific cases where the periodic repetition of a cellular module takes place in a single direction, homogeneous beam-like structures are successfully drawn up [5]. Civil engineering applications are also typical in this context, as in [6], where multiple-bay frames response is consistently reproduced by that of continuum models. The same assumptions are used in [7, 8], where tall steel-frame or reinforced-concrete building are dynamically analyzed through equivalent continua, in order to identify modal properties. Response to wind solicitations on tall buildings are evaluated in [9] where a calibration procedure permits to characterize the free dynamics. Extending the wind analysis to nonlinear field, homogeneous shear-beams are used to analyze the in-plane dynamic behavior of single [10] and coupled towers [11], even considering possible internal resonance [12] or in presence of base isolation system [13]. In case of three-dimensional frames, shear-shear-torsional equivalent beams are used in [14, 15, 16] to address aeroelastic effects on tower buildings, where systematic assessment of the bending to shear contribution to the deformation of the structure is carried out.

Drawing inspiration from [17], a shear-shear-torsional beam able to fulfill static nonlinear analysis of a tall building is proposed in [18, 19], and an elastic energy comparison between the rough and the detailed models is used to evaluate the main equivalent elastic parameters. On the same research line, buckling analysis is carried out in [20, 21], whereas improvements to the model are introduced in [22], where shear and flexural factors are used to consider possible floor deformation.

In [23], the free and forced dynamical features of a linear beam, derived from the same shear-shear-torsional model proposed in [18], are addressed, pointing out the special organization of the natural frequencies in triplets, which inevitably induce internal resonances, as well as the decomposition of normal modes to floor components, which are three, and axis-line components, governed by the wave number. Moving from [23], it appears compelling to give a first insight to nonlinear dynamics and stability of the proposed homogeneous model. In particular, in this paper, the nonlinear response to external resonant excitation is addressed in case of symmetric configuration of the building with respect to a principal axis of the floor. The specific shape and load produce an in-plane response, which is evaluated and analyzed applying the Multiple Scale Method [24] to the partial differential equation of motion of the beam. Then, in order to take into account the three-dimensional nature of the system, stability of the obtained in-plane solution to out-of-plane disturbances is addressed. The latter analysis is carried out consistently analyzing the linear variational equations, which turn out to be parametrically excited by a multi-frequency solicitation produced by the in-plane response itself. The nature of the in-plane response and its stability are then figured out on a numerical example, constituted by a 9-story building under uniformly distributed and resonant transverse load.

The paper is organized as follows: in Section 2., a brief recall to the main features of the homogeneous model is given; in Section 3. the in-plane nonlinear response is evaluated; in Section 4. the stability to out-of-plane disturbances is addressed; in Section 5. numeric results on a case-study are shown; finally, in Section 6. some conclusions are drawn.



2. The model

The model of shear-beam analyzed here is taken from [18], where details on its formulation are addressed. Here, for the sake of completeness, the model is briefly summarized, with specific considerations related to the pursued focus on nonlinear dynamic analysis: first, the hypothesis under which the model is formulated is recalled; then, the differential equations ruling the kinematic and equilibrium problems are presented; after that, the nonlinear elastic law, which comes from the homogenization procedure, is shown. With the sake of addressing a minimal model, where nonlinear contributions are only due to the elastic law, the equations of motions are obtained and then reduced to the case of symmetric building.

A modular tall building, i.e. realized by assembling many identical cells (stories) along the vertical direction, is considered (Fig. 1). As showed in [18, 23], the use of a homogeneous model of shear-beam to (coarsely) describe the building behavior is justified when the geometrical and mechanical properties assume values so that the following hypothesis is satisfied: the ratio between bending and shear deformation energy of the building is small enough. For instance, in case the building is constituted by a three-dimensional frame of equal stories, each of one realized by a horizontal rigid slab and N = 9 vertical columns, arranged in 3 equally spaced rows, that ratio is estimated as $r_{bs} \simeq 10.8 \left(\frac{\lambda_b}{\lambda_c}\right)^2$, where λ_b, λ_c are the building and column slenderness, respectively. In particular, indicating the inter-story height as h, the total height of the building as l = nh, the largest radius of inertia of the column cross-sections as ρ and a characteristic linear dimension of the story of the building as b, the column and building slenderness are defined as:

$$\lambda_c := \frac{h}{\rho}, \quad \lambda_b := \frac{l}{b} \tag{1}$$

As a consequence, in order to fulfill that specific energetic requirement, the columns should generally tend to be slender (large λ_c) and the building squat (small λ_b).



Fig. 1. Scheme of the n-story building. In red, the module of the j-th story.

Stated the aforementioned hypothesis, the homogeneous model considered here for the multi-story building is a shear-shear-torsional beam, constituted by an axis-line connected to the center of mass of infinite many, planar, rigid cross-sections (Fig. 2). Points of the axis-line perform displacement, whilst cross-sections perform twist. The current configuration at a generic time *t* is described by the axis-line displacement vector $\mathbf{u}(s,t) = u(s,t)\mathbf{a}_x + v(s,t)\mathbf{a}_y + w(s,t)\mathbf{a}_z$ and the cross-section twist vector $\boldsymbol{\theta}(s,t) = \theta(s,t)\mathbf{a}_x$, where $s \in [0, l]$ is the abscissa which runs along the axis line, *l* is its length and $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$ are unit orthogonal vectors of the canonical basis.

The strain components of the beam [17] are defined as:

$$\varepsilon(s,t) = u'(s,t)$$

$$\gamma_y(s,t) = v'(s,t)\cos(\theta(s,t)) + w'(s,t)\sin(\theta(s,t))$$

$$\gamma_z(s,t) = -v'(s,t)\sin(\theta(s,t)) + w'(s,t)\cos(\theta(s,t))$$

$$\kappa_t(s,t) = \theta'(s,t)$$
(2)

where ε is the axial strain, γ_y , γ_z are the shear strain components along \mathbf{a}_y and \mathbf{a}_z , respectively, κ_t is the torsion curvature and the prime stands for derivative with respect to s.

The normal and shear force components are N(s,t), $T_y(s,t)$, $T_z(s,t)$ and the torsion moment is M(s,t); they balance external transverse loads ($p_x(s,t)$, $p_y(s,t)$, $p_z(s,t)$) along the three canonical directions, and torsional couples c(s,t) about a_x . The equilibrium





Fig. 2. Scheme of the shear beam.

equations read:

$$N' + p_x = 0$$

$$(T_y \cos \theta - T_z \sin \theta)' + p_y = 0$$

$$(T_y \sin \theta + T_z \cos \theta)' + p_z = 0$$

$$M' + T_y(v' \sin \theta - w' \cos \theta) + T_z(v' \cos \theta + w' \sin \theta) + c = 0$$
(3)

where dependence on s, t is omitted for brevity.

Boundary conditions for the case of clamp at s = 0 (cross-section A) and free tip at s = l (cross-section B) are considered as well:

$$u_A = 0$$

$$v_A = 0$$

$$v_A = 0$$

$$\theta_A = 0$$
(4)

and:

$$N_B = P_x$$

$$T_{yB} \cos \theta - T_{zB} \sin \theta = P_y$$

$$T_{yB} \sin \theta + T_{zB} \cos \theta = P_z$$

$$M_B = C_x$$
(5)

where $(P_x(t)P_y(t), P_z(t), C_x(t))$ are assigned tip forces and couple.

The nonlinear elastic constitutive law is obtained from the homogenization process, which moves from writing the elastic potential energy of the assembly of columns and (assumed) rigid slabs (referred to as refined model) in terms of kinematic parameters of the shear-shear-torsional beam (referred to as coarse model), thorough kinematic maps (see [18]); at the end of the procedure, it reads: -/

$$\begin{pmatrix} N \\ T_y \\ T_z \\ M \end{pmatrix} = h \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & S_z & 0 & -z_S S_z \\ 0 & 0 & S_y & y_S S_y \\ 0 & -z_S S_z & y_S S_y & C \end{bmatrix} \begin{pmatrix} \varepsilon \\ \gamma_y \\ \gamma_z \\ \kappa_t \end{pmatrix} + \boldsymbol{f}_2(\boldsymbol{e}) + \boldsymbol{f}_2(\boldsymbol{e})$$
(6)

where:

$$\boldsymbol{f}_{2}\left(\boldsymbol{e}\right) \coloneqq \frac{6}{5}hD\begin{pmatrix} \frac{1}{2}\gamma_{y}^{2} + \frac{1}{2}\gamma_{z}^{2} + \frac{1}{2}\varrho_{G}^{2}\kappa_{t}^{2} - z_{E}\gamma_{y}\kappa_{t} + y_{E}\gamma_{z}\kappa_{t}\\ (\gamma_{y} - z_{E}\kappa_{t})\varepsilon\\ (\gamma_{z} + y_{E}\kappa_{t})\varepsilon\\ (\varrho_{G}^{2}\kappa_{t} + y_{E}\gamma_{z} - z_{E}\gamma_{y})\varepsilon \end{pmatrix}$$

$$(7)$$

and:

$$\mathbf{f}_{3}(\mathbf{e}) \coloneqq \frac{36}{35}hD \begin{pmatrix} 0 \\ \gamma_{y}(\gamma_{y}^{2} + \gamma_{z}^{2}) + \kappa_{t}[2\gamma_{y}\gamma_{z}y_{E} - z_{E}(3\gamma_{y}^{2} + \gamma_{z}^{2})] \\ + \kappa_{t}^{2}[\gamma_{y}(\varrho_{yy}^{2} + 3\varrho_{zz}^{2}) - 2\gamma_{z}\varrho_{yz}^{2}] - \kappa_{t}^{3}[\varrho_{yyz}^{3} + \varrho_{zzz}^{3}] \\ \gamma_{z}(\gamma_{y}^{2} + \gamma_{z}^{2}) + \kappa_{t}[y_{E}(\gamma_{y}^{2} + 3\gamma_{z}^{2}) - 2z_{E}\gamma_{y}\gamma_{z})] \\ + \kappa_{t}^{2}[\gamma_{z}(3\varrho_{yy}^{2} + \varrho_{zz}^{2}) - 2\gamma_{y}\varrho_{yz}^{2}] + \kappa_{t}^{3}[\varrho_{yyy}^{3} + \varrho_{yzz}^{3}] \\ (y_{E}\gamma_{z} - z_{E}\gamma_{y})(\gamma_{y}^{2} + \gamma_{z}^{2}) \\ + \kappa_{t}[\gamma_{y}^{2}(\varrho_{yy}^{2} + 3\varrho_{zz}^{2}) + \gamma_{z}^{2}(3\varrho_{yy}^{2} + \varrho_{zz}^{2}) - 4\gamma_{y}\gamma_{z}\varrho_{yz}^{2}] \\ - 3\kappa_{t}^{2}[\gamma_{y}(\varrho_{yyz}^{3} + \varrho_{zzz}^{3}) - \gamma_{z}(\varrho_{yyy}^{3} + \varrho_{yzz}^{3})] + \kappa_{t}^{3}\varrho_{GG}^{4} \end{pmatrix}$$
(8)

having indicated $e := (\varepsilon, \gamma_y, \gamma_z, \kappa_t)^T$. The parameters $D, S_y, S_z, C, y_S, z_S, y_E, z_E, \varrho_G, \varrho_{yy}, \varrho_{zz}, \varrho_{yz}, \varrho_{yyz}, \varrho_{yzz}, \varrho_{zzz}, \varrho_{GG}$ are related to the shape and elastic properties of the frame. Specifically, D is the total extensional stiffness, S_y , S_z are the shear stiffness in the two orthogonal directions,



respectively, *C* is the torsion stiffness, y_S , z_S are the coordinates of the shear center **S** of the cross-section, y_E , z_E are the coordinates of the extensional center **E** of the cross-section (the centroid of extensional stiffness of the columns), the ϱ 's are inertia radii of the extensional stiffness, of second, third and fourth order, evaluated with respect to the centroid **G**. The expression of all the involved coefficients is given in Appendix A.

In the specific case that is addresses in this paper, no axial loads are considered ($p_x = 0$, $P_x = 0$), since only transversal loads are applied to the beam. As a consequence, from Eq. (3-a) and Eq. (5-a), it is $N \equiv 0$ and, from Eq. (6), it turns out that:

$$\varepsilon = -\frac{3}{5} \left(\gamma_y^2 + \gamma_z^2 + \varrho_G^2 \kappa_t^2 - 2z_E \gamma_y \kappa_t + 2y_E \gamma_z \kappa_t \right) \tag{9}$$

Eq. (9) entails that ε is a second order variable, slave of the other strain components: this means that a uniform stretch of the columns is required to induce a normal force on them, which balances (and vanishes) the resulting normal force induced during transversal displacement or twist of the slabs. Substitution of Eq. (9) in Eq. (6) produces a statically condensed version of the following constitutive law, which is strictly cubic (i.e., no quadratic terms in the master strain components appear):

/ \

$$\begin{pmatrix} T_y \\ T_z \\ M \end{pmatrix} = h \begin{bmatrix} S_z & 0 & -z_S S_z \\ 0 & S_y & y_S S_y \\ -z_S S_z & y_S S_y & C \end{bmatrix} \begin{pmatrix} \gamma_y \\ \gamma_z \\ \kappa_t \end{pmatrix} + g(e)$$
(10)

where:

$$g(e) = \frac{18}{175}Dh \begin{pmatrix} 3\gamma_y(\gamma_y^2 + \gamma_z^2) + 3\kappa_t [2y_E\gamma_y\gamma_z - z_E(3\gamma_y^2 + \gamma_z^2)] \\ +\kappa_t [14z_E(y_E\gamma_z - z_E\gamma_y) \\ +\gamma_y(10\varrho_{yy}^2 - 7\varrho_G^2 + 30\varrho_{zz}^2) - 20\gamma_z \varrho_{yz}^2] \\ +\kappa_t^3 [7z_E \varrho_G^2 - 10(\varrho_{yyz}^3 + \varrho_{zzz}^3)] \\ \hline 3\gamma_z(\gamma_y^2 + \gamma_z^2) - 3\kappa_t [2z_E\gamma_y\gamma_z - y_E(\gamma_y^2 + 3\gamma_z^2)] \\ +\kappa_t^2 [\gamma_y(14y_Ez_E - 20\varrho_{yz}^2) \\ +\gamma_z(10\varrho_{zz}^2 - 14y_E^2 - 7\varrho_G^2)] \\ \hline -\kappa_t^3 [7y_E \varrho_G^2 - 10(\varrho_{yyy}^3 + \varrho_{yzz}^3)] \\ \hline 3(y_E\gamma_z - z_E\gamma_y)(\gamma_y^2 + \gamma_z^2) \\ +\kappa_t [\gamma_y(28y_Ez_E\gamma_z + \gamma_y(10\varrho_{yy}^2 - 14z_E^2 - 7\varrho_G^2)) \\ -\gamma_z(40\gamma_y \varrho_{yz}^2 + \gamma_z(14y_E^2 + 7\varrho_G^2 - 30\varrho_{yy}^2)) \\ +10\varrho_{zz}^2(3\gamma_y^2 + \gamma_z^2)] \\ +3\kappa_t^2 [7\varrho_G^2(z_E\gamma_y - y_E\gamma_z) + 10\gamma_z(\varrho_{yyy}^3 + \varrho_{yzz}^3) \\ -10\gamma_y(\varrho_{yyz}^3 + \varrho_{zzz}^3)] \\ +\kappa_t^3 [10\varrho_{GG}^4 - 7\varrho_G^4] \end{pmatrix}$$
(11)

In order to consider the simplest possible model, even featuring the stronger nonlinear behavior, the following approximation is carried out: linearized versions of kinematic and equilibrium equations are used, whereas the only nonlinear contributions are given by the condensed constitutive law Eq. (10) (details on the order of magnitude of the nonlinear terms are given in [18]). Under these assumptions, Eqs. (2) and (3) become:

and

$$\begin{aligned} \gamma_y &= v' \\ \gamma_z &= w' \\ \kappa_t &= \theta' \end{aligned} \tag{12}$$

$$T'_{y} + p_{y} - m\dot{v} - c_{v}\dot{v} = 0$$

$$T'_{z} + p_{z} - m\ddot{w} - c_{w}\dot{w} = 0$$

$$M' + c - I_{G}\ddot{\theta} - c_{\theta}\dot{\theta} = 0$$
(13)

respectively, with boundary conditions

 $v_A = 0$ $w_A = 0$ $\theta_A = 0$ (14)

and

$$T_{yB} = P_{yB}$$

$$T_{zB} = P_{zB}$$

$$M_B = C_B$$
(15)

where, in Eqs. (13), d'Alembert formula is used for expressing inertia and damping force, with m the linear mass density of the beam, I_G the mass inertia moment about the centroid **G** of the cross-section of the beam, c_y , c_z , c_θ the damping coefficients. It is worth noticing that m and I_G are obtained as the total mass and mass inertia moment of the module of the building, divided by the inter-story height h.

Substituting Eqs. (12) in (10), and then the latter in (13), the equations of motion are obtained. Combining them with boundary conditions (14) and (15), they turn out to be:

$$-M\ddot{u} - C\dot{u} + Ku'' + n_3(u, u, u) = -p$$

$$u_A = 0$$

$$Ku'_B + b_3(u_B, u_B, u_B) = P$$
(16)



where the mass, damping, (algebraic part of) stiffness matrices and load column matrix are:

$$\boldsymbol{M} := \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_G \end{bmatrix}, \quad \boldsymbol{C} := \begin{bmatrix} c_y & 0 & 0 \\ 0 & c_z & 0 \\ 0 & 0 & c_\theta \end{bmatrix}, \quad \boldsymbol{K} := h \begin{bmatrix} S_z & 0 & -z_S S_z \\ 0 & S_y & y_S S_y \\ -z_S S_z & y_S S_y & C \end{bmatrix},$$

$$\boldsymbol{p} := \begin{pmatrix} p_y \\ p_z \\ c \end{pmatrix}, \qquad \boldsymbol{P} := \begin{pmatrix} P_{yB} \\ P_{zB} \\ C_B \end{pmatrix}, \qquad \boldsymbol{u} := \begin{pmatrix} v \\ w \\ \theta \end{pmatrix}$$
(17)

The expression for $b_3(u, u, u)$ is easy obtained substituting Eqs. (12) in Eq. (11); then, it can be evaluated at section *B* to get the nonlinear term in Eq. (16-c). Moreover, *s*-differentiation of $b_3(u, u, u)$ provides $n_3(u, u, u)$, for Eq. (16-a).

In case of proportional damping, the free linear dynamics of the system is ruled from Eqs. (16) after dropping the nonlinear terms n_3, b_3 , the forcing terms p, P and the damping term $C\dot{u}$. It is worth noticing that, due to the specific form of the obtained system, the free dynamics has peculiar features, as shown in detail in [23]. Specifically, using the variable separation and assuming the solution $u(s, t) = \phi(s) \exp(i\omega t)$, the differential boundary value problem that rules free linear oscillations is:

$$\boldsymbol{K}\boldsymbol{\phi}^{\prime\prime} + \omega^2 \boldsymbol{M}\boldsymbol{\phi} = \boldsymbol{0} \tag{18}$$

$$\phi_A = \mathbf{0} \tag{19}$$

$$K\phi'_B = 0 \tag{20}$$

A trial solution for Eq. (18) is:

$$\phi(s) = a \sin\left(\frac{k\pi}{2l}s\right), \qquad k = 1, 3, 5, \dots$$
(21)

where a is a column vector of three elements, which collects amplitudes of the components of motion (*v*-translation, *w*-translation and twist θ). Substituting Eq. (21) in (18) gives the following algebraic eigenvalue problem:

$$(K - \lambda M) a = 0 \tag{22}$$

with:

$$\lambda := \frac{4l^2}{k^2 \pi^2} \omega^2 \tag{23}$$

which provides the following characteristic equation:

$$\det\left(\boldsymbol{K}-\lambda\boldsymbol{M}\right)=0\tag{24}$$

Due to the positive definiteness of matrices K, M, three positive characteristic roots λ_j (j = 1, 2, 3) are obtained and, consequently, the natural frequencies are evaluated from Eq. (23) for any chosen k:

$$\omega_{kj} = \frac{k\pi}{2l} \sqrt{\lambda_j} \qquad k = 1, 3, 5, \dots, \quad j = 1, 2, 3$$
(25)

By summarizing, the system admits a triplet of frequencies for each wave-number k, (or semi-wavelength 2l/k). Once determined the first triplet of frequencies ($\omega_{11}, \omega_{12}, \omega_{13}$) by means of the three solutions of the associated characteristic equation (24) and k = 1, the successive natural frequencies are simply evaluated as multiple of the first ones:

$$\omega_{kj} = \omega_{11}, \omega_{12}, \omega_{13}; \ 3\omega_{11}, 3\omega_{12}, 3\omega_{13}; \ 5\omega_{11}, 5\omega_{12}, 5\omega_{13}; \ \dots$$
(26)

Consistently, the modes can be decomposed in axis-line and cross-section ones: the latter are the same for any triplet of frequency, whilst the former change their wave-number at each triplet. This means that, independently of the value of k, the eigenvector a_j associated to each eigenvalue λ_j is found from Eq. (22), for j = 1, 2, 3; it describes the cross-section natural mode. In other words, there exist three amplitude vectors a_1 , a_2 , a_3 , which repeat themselves for any wave-number $k = 1, 3, 5, \ldots$; they represent the cross-section modal components and contribute to the beam natural modes as follows:

$$\phi_{kj}(s) = a_j \sin\left(\frac{k\pi}{2l}s\right) \text{ for } k = 1, 3, 5, \dots, j = 1, 2, 3$$
(27)

The natural frequency organization given by Eq. (26) produces internal resonance conditions.

3. Nonlinear in-plane response for symmetric buildings

Here, the case of a symmetric layout of the columns about the \mathbf{a}_y axis is considered. In this case, Eq. (22) admits solution $\mathbf{a}_1 = (1, 0, 0)^T$ and the corresponding natural frequencies are $\omega_{k1} = \frac{k\pi}{2l} \sqrt{\frac{hS_z}{m}}$, with $k = 1, 3, 5, \ldots$. Moreover, only loads which respect the same symmetry conditions are assumed to be applied to the building. Therefore, attention is paid on vibrations occurring in the $(\mathbf{a}_x, \mathbf{a}_y)$ plane, as caused by a resonant external excitation; however, stability of the in-plane solution to out-of-plane disturbances is analyzed as well.

Consistently with the notation adopted in [23], the in-plane frequencies are denoted by $\omega_{k1} = k\omega_{11}$, k = 1, 3, 5, ..., and the out-of-plane frequencies by $\omega_{kj} = k\omega_{1j}$, k = 1, 3, 5, ..., j = 2, 3.

When the building oscillates in the considered plane of symmetry, the equations of motion (16) reduce to:

$$h\left[S_{z}v'(s,t) + \frac{54}{175}Dv'^{3}(s,t)\right]' - m\ddot{v}(s,t) - c_{y}\dot{v}(s,t) + \alpha p_{y}(s,t) = 0$$

$$v_{A} = 0$$

$$h\left[S_{z}v'_{B} + \frac{54}{175}Dv'^{3}_{B}\right] = P_{yB}(t)$$
(28)



where the load multiplier α is introduced.

It is considered an excitation constituted by distributed forces, $p_y(s,t) = p_y(s) \cos \Omega t$ (while $P_{yB}(t) \equiv 0$).

Nondimensional independent and dependent variables are defined as follows:

$$\tilde{t} := \omega t, \quad \tilde{s} := \frac{s}{l}, \quad \tilde{v} := \frac{v}{l}$$
(29)

Consequently it turns out that

$$\frac{d}{dt} = \omega \frac{d}{d\bar{t}}, \quad \frac{d}{ds} = \frac{1}{l} \frac{d}{d\bar{s}}$$
(30)

where $\omega := \omega_{11}$. After substitution of Eq. (29) and (30) in Eq. (28), the following system is obtained:

$$-\ddot{v} - 2\xi \dot{v} + \kappa \tilde{v}'' + \frac{\eta}{3} (\tilde{v}^{\prime 3})' + \alpha p \cos(\tilde{\Omega} \tilde{t}) = 0$$

$$\tilde{v}_A = 0$$

$$\kappa \tilde{v}_B' + \frac{\eta}{3} \tilde{v}_B'^3 = 0$$
(31)

where the following nondimensional parameters are defined:

$$\begin{aligned} \xi &:= \frac{c_y}{m\omega}, \qquad \kappa := \frac{hS_y}{m\omega^2 l^2}, \qquad \eta := \frac{162hD}{175m\omega^2 l^2}, \\ p &:= \frac{p_y}{m\omega^2 l}, \quad \tilde{\Omega} := \frac{\Omega}{\omega}, \qquad \tilde{\omega}_{kj} := \frac{\omega_{kj}}{\omega} \end{aligned}$$
(32)

and now prime and dots stand for differentiation with respect to \tilde{s} and \tilde{t} , respectively. Furthermore, the subscript ()_A indicates evaluation at $\tilde{s} = 0$ and ()_B at $\tilde{s} = 1$. The symbol tilde is omitted below.

The Multiple Scale Method is used and the case of primary resonance with the in-plane frequency of the first triplet is considered, i.e.:

$$\Omega = 1 + \epsilon \sigma_e \tag{33}$$

with $0 < \epsilon \ll 1$ and σ_e a order-1 detuning (the index e denoting external).

By ordering damping and forces as $\xi \to \epsilon \xi$, $\alpha \to \epsilon^{3/2} \alpha$, and rescaling the displacement as $v \to \epsilon^{1/2} v$, Eqs. (31) read:

$$-\ddot{v} + \kappa v'' - \epsilon \left[2\xi \dot{v} - \frac{\eta}{3} (v'^3)' - \alpha p \cos(\Omega t)\right] = 0$$

$$v_A = 0$$

$$\kappa v'_B + \epsilon \frac{\eta}{3} v'^3_B = 0$$
(34)

Introducing independent time-scales $t_0 = t$, $t_1 = \epsilon t$, ..., and expanding the response as:

$$v = v_0(s, t_0, t_1, \ldots) + \epsilon v_1(s, t_0, t_1, \ldots) + \ldots$$
(35)

the following perturbation equations are obtained:

$$\partial_0^2 v_0 - \kappa v_0'' = 0$$

$$v_{0A} = 0$$

$$\kappa v_{0B}' = 0$$
(36)

at order ϵ^0 and

$$\partial_0^2 v_1 - \kappa v_1'' = -2\partial_0 \partial_1 v_0 - 2\xi \partial_0 v_0 + \frac{\eta}{3} (v_0^3)' + \alpha p \cos(\Omega t)$$

$$v_{1A} = 0$$

$$\kappa v_{1B}' = -\frac{\eta}{3} v_{0B}'^3$$
(37)

at order ϵ^1 , in which $\partial_j^n = \frac{\partial^n}{\partial t_i^n}$.

Due to the internal resonance between the first mode (k = 1, directly excited, $\omega_{11} = 1$) and the second mode (k = 3, companion, $\omega_{31} = 3$), a generating solution with two components for Eq. (36) is considered, other resonances appearing at orders higher than 3:

$$v_0 = A_1(t_1)\phi_{11}(s)e^{i\omega_{11}t_0} + A_3(t_1)\phi_{31}(s)e^{i\omega_{31}t_0} + cc$$
(38)

where A_1 , A_3 are complex amplitudes, $\phi_{11}(s) = \sin\left(\frac{\pi s}{2}\right)$, $\phi_{31}(s) = \sin\left(\frac{3\pi s}{2}\right)$ and cc stands for complex conjugate. When Eq. (38) is substituted in Eq. (37), this latter reads:

$$\begin{aligned} \partial_{0}^{2} v_{1} - \kappa v_{1}^{\prime\prime} &= \left\{ -2i\omega_{11}\phi_{11}\partial_{1}A_{1} - 2i\xi\omega_{11}\phi_{11}A_{1} + \eta[3\phi_{11}^{\prime\prime}\phi_{11}^{\prime\prime}A_{1}^{2}\bar{A}_{1} \right. \\ &+ \left(2\phi_{31}^{\prime2}\phi_{11}^{\prime\prime\prime} + 4\phi_{11}^{\prime}\phi_{31}^{\prime\prime}\phi_{31}^{\prime\prime} \right)A_{1}A_{3}\bar{A}_{3} + \left(\phi_{11}^{\prime2}\phi_{31}^{\prime\prime\prime} \\ &+ 2\phi_{31}^{\prime}\phi_{11}^{\prime\prime}\mu_{11}^{\prime\prime\prime} \right)A_{3}\bar{A}_{1}^{2}] + \frac{\alpha p}{2}e^{i\sigma_{e}t_{1}} \right\}e^{i\omega_{11}t_{0}} \\ &+ \left\{ -2i\omega_{31}\phi_{31}\partial_{1}A_{3} - 2i\xi\omega_{31}\phi_{31}A_{3} \\ &+ \eta[\phi_{11}^{\prime2}\phi_{11}^{\prime\prime}A_{1}^{3} + \left(2\phi_{11}^{\prime2}\phi_{31}^{\prime\prime\prime} + 4\phi_{11}^{\prime}\phi_{31}^{\prime}\phi_{11}^{\prime\prime\prime} \right)A_{1}A_{3}\bar{A}_{1} \\ &+ 3\phi_{31}^{\prime2}\phi_{31}^{\prime\prime}A_{3}^{2}\bar{A}_{3} \right] \right\}e^{i\omega_{31}t_{0}} + NRT \end{aligned}$$

$$(39)$$



where the overbar indicates complex conjugate and NRT stands for non resonant terms. Solvability requires that: (i) the coefficients of $e^{i\omega_{11}t_0}$ are orthogonal to $\phi_{11}(s)$, and (ii) the coefficients of $e^{3i\omega_{11}t_0}$ are orthogonal to $\phi_{31}(s)$ [24]. From these two orthogonality conditions, two amplitude modulation equations in the unknown amplitudes arise: once one comes back to the original and not rescaled variables, they read:

$$\dot{A}_{1} = c_{1}A_{1} + ic_{11\bar{1}}A_{1}^{2}\bar{A}_{1} + ic_{\bar{1}\bar{1}\bar{3}}\bar{A}_{1}^{2}A_{3} + ic_{13\bar{3}}A_{1}A_{3}\bar{A}_{3} + \alpha ic_{0}e^{i\sigma_{e}t}$$

$$\dot{A}_{3} = c_{3}A_{3} + ic_{111}A_{1}^{3} + ic_{1\bar{1}\bar{3}}A_{1}\bar{A}_{1}A_{3} + ic_{33\bar{3}}A_{3}^{2}\bar{A}_{3}$$
(40)

where:

$$c_{1} = -\xi, \quad c_{11\bar{1}} = \frac{3\eta\pi^{4}}{128\omega_{11}}, \quad c_{\bar{1}\bar{1}\bar{3}} = \frac{3\eta\pi^{4}}{128\omega_{11}}, \quad c_{13\bar{3}} = \frac{9\eta\pi^{4}}{32\omega_{11}},$$

$$c_{3} = -\xi, \quad c_{111} = \frac{\eta\pi^{4}}{128\omega_{31}}, \quad c_{1\bar{1}\bar{3}} = \frac{9\eta\pi^{4}}{32\omega_{31}}, \quad c_{33\bar{3}} = \frac{243\eta\pi^{4}}{128\omega_{31}}$$

$$c_{0} = -\frac{1}{2\omega_{11}} \int_{0}^{1} p(s)\phi_{11}(s)ds$$
(41)

The complex Eqs. (40) are equivalent to four non-autonomous real equations in amplitude and phase, obtained by putting $A_1 =$ $\frac{1}{2}q_1(t) \exp(i\varphi_1(t))$, $A_3 = \frac{1}{2}q_3(t) \exp(i\varphi_3(t))$ and separating real and imaginary parts. They are more conveniently rewritten in terms of the phase-differences:

$$\gamma_e := \sigma_e t - \varphi_1$$

$$\gamma_i := 3\varphi_1 - \varphi_3$$
(42)

in which γ_e accounts for the (detuned) external resonance, and γ_i for the (perfectly tuned) internal resonance. Using the definitions above, Eqs. (40) become:

$$\dot{q}_{1} = c_{1}q_{1} - 2c_{0}\alpha \sin\gamma_{e} + \frac{1}{4}c_{\bar{1}\bar{1}\bar{3}}q_{1}^{2}q_{3}\sin\gamma_{i}$$

$$\dot{q}_{3} = c_{3}q_{3} - \frac{1}{4}c_{111}q_{1}^{3}\sin\gamma_{i}$$

$$q_{1}\dot{\gamma}_{e} = q_{1}\sigma_{e} - \frac{1}{4}c_{11\bar{1}}q_{1}^{3} - \frac{1}{4}c_{13\bar{3}}q_{1}q_{3}^{2} - 2c_{0}\alpha\cos\gamma_{e} - \frac{1}{4}c_{\bar{1}\bar{1}\bar{3}}q_{1}^{2}q_{3}\cos\gamma_{i}$$

$$q_{1}q_{3}\dot{\gamma}_{i} = \left(\frac{3}{4}c_{11\bar{1}} - \frac{1}{4}c_{1\bar{1}\bar{3}}\right)q_{1}^{3}q_{3} + \left(\frac{3}{4}c_{13\bar{3}} - \frac{1}{4}c_{33\bar{3}}\right)q_{1}q_{3}^{3} + 6c_{0}\alpha q_{3}\cos\gamma_{e} - \frac{1}{4}c_{111}q_{1}^{4}\cos\gamma_{i} + \frac{3}{4}c_{\bar{1}\bar{1}\bar{3}}q_{1}^{2}q_{3}^{2}\cos\gamma_{i}$$

$$(43)$$

The set of four amplitudes and phases are denoted as $\boldsymbol{\gamma} := (q_1, q_3, \gamma_e, \gamma_i)^T$. In the new variables, by naming $\Phi_1 := \omega_{11}t + \varphi_1$, $\Phi_3 := 3\omega_{11}t + \varphi_3$ the total phases, they turn out to be:

$$\Phi_1 = \omega_{11}t + \sigma_e t - \gamma_e = \Omega t - \gamma_e$$

$$\Phi_3 = 3\omega_{11}t + 3\varphi_1 - \gamma_i = 3\omega_{11}t + 3\sigma_e t - 3\gamma_e - \gamma_i = 3(\Omega t - \gamma_e) - \gamma_i$$
(44)

so that, remembering the expressions for $\phi_{11}(s)$ and $\phi_{31}(s)$, at the leading order:

$$v(s,t) = \frac{1}{2}q_1 \sin\left(\frac{\pi s}{2}\right) e^{i(\Omega t - \gamma_e)} + \frac{1}{2}q_3 \sin\left(\frac{3\pi s}{2}\right) e^{i(3\Omega t - 3\gamma_e - \gamma_i)} + cc$$

$$= q_1 \sin\left(\frac{\pi s}{2}\right) \cos(\Omega t - \gamma_e) + q_3 \sin\left(\frac{3\pi s}{2}\right) \cos(3\Omega t - 3\gamma_e - \gamma_i)$$
(45)

The equilibrium points $\dot{\gamma} = 0$ of the dynamical system (43) are periodic motion for the building, which will be denoted by $v_e(s,t)$. In them, amplitudes and phases depend on the intensity of the load via $q_1 = q_1(\alpha)$, $q_3 = q_3(\alpha)$, $\gamma_e = \gamma_e(\alpha)$, $\gamma_i = \gamma_i(\alpha)$.

It should be noticed that no uncoupled solutions are admitted, since the external force excites the k = 1 mode (Eq. (43-a) and this, in turn, excites the k = 3 mode, by the way of the q_1^3 -term (Eq. (43-b). Equilibrium solutions can only be evaluated numerically.

4. Stability of the in-plane response to out-of-plane disturbances

4.1 The variational equation

To analyze stability of the planar periodic response (45), incremental variables \hat{v} , \hat{w} , $\hat{ heta}$, are introduced so that $v = v_e + \hat{v}$, $w = \hat{w}$, $\theta = \hat{\theta}$ and the equations of motion are linearized around the increments. Such linear equations, of course, are unable to provide quantitative information on the evolution of perturbation; however, they give a correct qualitative answer to the question of stability. Accordingly, the periodic motion (45) is stable when perturbations tend to zero, and unstable when they diverge. A more accurate description of the postcritical behavior would require obtaining nonlinear amplitude modulation equations including all possible modes, planar and non-planar, involved in the instability phenomenon, but that is out of the scope of this paper. Consistently with Eqs. (29) and (30) and defining $\tilde{w} = \frac{w}{l}$, $\tilde{\theta} = \theta$, the following nondimensional parameters are introduced:

$$\kappa_{w} = \kappa \frac{S_{y}}{S_{z}}, \qquad \kappa_{w\theta} = \kappa \frac{S_{y}y_{S}}{S_{z}l}, \qquad \kappa_{\theta w} = \kappa \frac{S_{y}y_{S}ml}{S_{z}I_{G}}, \qquad \kappa_{\theta} = \kappa \frac{Cm}{S_{z}I_{G}},
\eta_{w} = \kappa \frac{54D}{175S_{z}}, \qquad \eta_{w\theta} = \kappa \frac{54Dy_{E}}{175I_{S}z}, \qquad \eta_{\theta w} = \kappa \frac{54Dlmy_{E}}{175I_{G}S_{z}}
\eta_{\theta} = \kappa \frac{Dm}{S_{z}I_{G}} \left(-\frac{18\rho_{G}^{2}}{25} + \frac{36\rho_{yy}^{2}}{25} + \frac{108\rho_{zz}^{2}}{35} \right), \qquad \xi_{w} = \frac{c_{z}}{m\omega}, \qquad \xi_{\theta} = \frac{c_{\theta}}{I_{G}\omega}$$
(46)

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and, accounting for the symmetry of the systems, which entails $z_S = z_E = 0$, the linearized (variational) equations in the out-ofplane increments, with hat and tilde symbol omitted, read

$$-\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \ddot{v} \\ \ddot{w} \\ \ddot{\theta} \end{pmatrix} - \begin{bmatrix} 2\xi & 0 & 0 \\ 0 & 2\xi_z & 0 \\ 0 & 0 & 2\xi_\theta \end{bmatrix} \begin{pmatrix} \dot{v} \\ \dot{w} \\ \dot{\theta} \end{pmatrix} + \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa_w & \kappa_{w\theta} \\ 0 & \kappa_{\theta w} & \kappa_{\theta} \end{bmatrix} \begin{pmatrix} v'' \\ w'' \\ \theta'' \end{pmatrix} + \begin{bmatrix} v'^2_e(s,t) \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_w & \eta_{w\theta} \\ 0 & \eta_{\theta w} & \eta_\theta \end{pmatrix} \begin{pmatrix} v' \\ w' \\ \theta' \end{pmatrix} \end{bmatrix}' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
(47)

Similarly, the boundary conditions read:

$$v_A = 0, \qquad w_A = 0, \qquad \theta_A = 0$$
 (48)

and:

$$\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa_w & \kappa_{w\theta} \\ 0 & \kappa_{\theta w} & \kappa_{\theta} \end{bmatrix} \begin{pmatrix} v'_B \\ w'_B \\ \theta'_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
(49)

in which $v_e^{\prime 2}(l,t) = 0$ is accounted for. It is worth noticing that the stiffness algebraic operator in Eq. (47) and (49) appears as nonsymmetric, even if the system is Hamiltonian (excluding damping and external force), only as a consequence of the division of the second and third equations by the relevant inertial coefficients, m and I_G , respectively, and the used definition of nondimensional variables.

Since $v_e(s, t)$ is a known function of space and time, Eqs. (47)-(49) represent a non-autonomous linear system, with periodic in time coefficients.

By accounting for the expression (45), system (47)-(49) can be rewritten as follows:

$$-\ddot{\boldsymbol{u}} - \boldsymbol{\Xi}\dot{\boldsymbol{u}} + \boldsymbol{\Theta}\boldsymbol{u}'' + \left[\left(\sum_{r=0,2,4,6} f_r\left(s;\boldsymbol{\gamma}\right) e^{ir\Omega t} + cc \right) \boldsymbol{N}\boldsymbol{u}' \right]' = \boldsymbol{0}$$

$$\boldsymbol{u}_A = \boldsymbol{0}$$

$$\boldsymbol{\Theta}\boldsymbol{u}_B' = \boldsymbol{0}$$
(50)

where Ξ and Θ are the 3 × 3 damping and stiffness matrices, and N the 3 × 3 parametric excitation matrix, all defined by comparison with Eqs. (47); moreover:

$$f_{0}(s;\boldsymbol{\gamma}) := \left(\frac{\pi}{4}\right)^{2} q_{1}^{2} \cos^{2}\left(\frac{\pi s}{2}\right) + \left(\frac{3\pi}{4}\right)^{2} q_{3}^{2} \cos^{2}\left(\frac{3\pi s}{2}\right)$$

$$f_{2}(s;\boldsymbol{\gamma}) := \left(\frac{\pi}{4}\right)^{2} q_{1}^{2} e^{-2i\gamma_{e}} \cos^{2}\left(\frac{\pi s}{2}\right)$$

$$+ \frac{3\pi^{2}}{8} q_{1} q_{3} e^{-i[2\gamma_{e}+\gamma_{i}]} \cos\left(\frac{\pi s}{2}\right) \cos\left(\frac{3\pi s}{2}\right)$$

$$f_{4}(s;\boldsymbol{\gamma}) := \frac{3\pi^{2}}{8} q_{1} q_{3} e^{-i[4\gamma_{e}+\gamma_{i}]} \cos\left(\frac{\pi s}{2}\right) \cos\left(\frac{3\pi s}{2}\right)$$

$$f_{6}(s;\boldsymbol{\gamma}) := \left(\frac{3\pi}{4}\right)^{2} q_{3}^{2} e^{-i[6\gamma_{e}+2\gamma_{i}]} \cos^{2}\left(\frac{3\pi s}{2}\right)$$
(51)

are (complex) known function of space, depending on amplitude and phases of the planar solution, which describe the magnitude and distribution of the parametric excitation. It is worth noting that a multi-frequency parametric excitation appears, as generated by the combination of the two harmonics Ω and 3Ω which are contained in the periodic planar motion.

4.2 Perturbation analysis

To solve Eqs. (50), again use the Multiple Scale Method is made. Since the parametric excitation depends on the squared amplitudes, the rescaling $f_r \rightarrow \epsilon f_r$, together with $\Xi \rightarrow \epsilon \Xi$ is used. Moreover, time-scales $t_0 = t$, $t_1 = \epsilon t$,... are introduced and the displacements are expanded as:

$$\boldsymbol{u} = \boldsymbol{u}_0 + \epsilon \boldsymbol{u}_1 \tag{52}$$

thus obtaining the following perturbation equations:

 $\Theta \boldsymbol{u}_0^{\prime\prime} - \partial_0^2 \boldsymbol{u}_0 = \boldsymbol{0}$ $\boldsymbol{u}_{0A} = \boldsymbol{0}$ $\Theta \boldsymbol{u}_{0B}^{\prime} = \boldsymbol{0}$ (53)

after collecting terms at order ϵ^0 , and:

$$\Theta \boldsymbol{u}_{1}^{\prime\prime} - \partial_{0}^{2} \boldsymbol{u}_{1} = 2\partial_{0}\partial_{1}\boldsymbol{u}_{0} + \Xi \partial_{0}\boldsymbol{u}_{0} - \left[\left(\sum_{r=0,2,4,6} f_{r}(s;\boldsymbol{\gamma})e^{ir\Omega t_{0}} + cc \right) \boldsymbol{N}\boldsymbol{u}_{0}^{\prime} \right]^{\prime}$$

$$\boldsymbol{u}_{1A} = \boldsymbol{0}$$

$$\Theta \boldsymbol{u}_{1B}^{\prime} = \boldsymbol{0}$$
(54)

at order $\epsilon^1.$

In expressing the generating solution to Eqs. (53), all modes which are involved in the parametric resonance phenomenon are included. Since the excitation is multifrequent, and internal resonances exist, several modes can be involved in the response, rendering the analysis very difficult. To limit the difficulties, the following hypotheses are introduced:



- 1. Only modes which are in principal parametric resonance are considered. This occurrence is verified when the excitation frequency, $r\Omega$, (r = 2, 4, 6) is double of the natural frequency ω_{kj} . Since $\Omega \simeq \omega_{11}$, this happens when $\frac{\omega_{kj}}{\omega_{11}} = 1, 2, 3$. Note that this requirement concerns the frequency, not the mode, so that any k can (in principle) be associated to the parametrically excited mode.
- 2. Due to the multifrequency excitation, more than a single out-of-plane mode can be parametrically excited; however, to limit the possible cases, only one out-of-plane mode involved in parametric resonance is considered.

According to the previous discussion, the generating solution is:

$$\boldsymbol{u}_0 = B(t_1)\boldsymbol{a}_j \sin\left(\frac{k\pi s}{2}\right) e^{i\omega_{kj}t_0} + cc$$
(55)

where $B(t_1)$ is a complex modulating amplitude, the column matrix a_j is a (real, normalized) natural mode ($(\Theta - \lambda_j I)a_j = 0$, where I is the identity matrix), and the associated natural frequency satisfies:

$$2\omega_{kj} = r\omega_{11} + \epsilon\sigma_p = r\Omega + \epsilon \left(\sigma_p - r\sigma_e\right) \qquad r = 2, 4, 6$$
(56)

in which σ_p is a new detuning (*p* remembering parametric excitation) and where Eq. (33) is accounted for.

By substituting the generating solution into the ϵ -order field perturbation equations (54-a), this latter becomes:

$$\Theta \boldsymbol{u}_{1}^{\prime\prime} - \partial_{0}^{2} \boldsymbol{u}_{1} = i\omega_{kj} [2\boldsymbol{a}_{j}\partial_{1}B + \Xi \boldsymbol{a}_{j}B]e^{i\omega_{kj}t_{0}} \sin\left(\frac{k\pi s}{2}\right) + cc$$

$$-\frac{k\pi}{2} \left[\sum_{r=0,2,4,6} (f_{r}(s;\boldsymbol{\gamma})e^{ir\Omega t_{0}} + cc)(Be^{i\omega_{kj}t_{0}} + cc)\cos\left(\frac{k\pi s}{2}\right)\right]^{\prime} \boldsymbol{N}\boldsymbol{a}_{j}$$
(57)

Among terms of the sum, just one (for the hypotheses introduced) is resonant, namely that corresponding to the value of r which satisfies condition (56). Therefore, the previous equation also reads:

$$\Theta \boldsymbol{u}_{1}^{\prime\prime} - \partial_{0}^{2} \boldsymbol{u}_{1} = i\omega_{kj} [2\boldsymbol{a}_{j}\partial_{1}B + \Xi \boldsymbol{a}_{j}B]e^{i\omega_{kj}t_{0}} \sin\left(\frac{k\pi s}{2}\right) - \left\{\frac{k\pi}{2} \boldsymbol{N}\boldsymbol{a}_{j} \left[2Bf_{0} + \bar{B}f_{r}e^{it_{0}(r\sigma_{e}-\sigma_{p})}\right] \cos\left(\frac{k\pi s}{2}\right)\right\}^{\prime} e^{i\omega_{kj}t_{0}} + cc + NRT$$
(58)

Solvability of Eq. (58) requires that the right hand member is orthogonal to $\check{a}_j \sin\left(\frac{k\pi s}{2}\right)$, where \check{a}_j is solution to $(\Theta^T - \lambda_j I)\check{a}_j$, i.e.:

$$-i\omega_{kj}\check{a}_{j}^{T}[2a_{j}\partial_{1}B + \Xi a_{j}B] \int_{0}^{1} \sin^{2}\left(\frac{k\pi s}{2}\right) ds$$

$$+ \frac{k\pi}{2}\check{a}_{j}^{T}Na_{j}\left[2B\int_{0}^{1}\left(f_{0}\cos\left(\frac{k\pi s}{2}\right)\right)'\sin\left(\frac{k\pi s}{2}\right)ds$$

$$+ \bar{B}e^{it_{0}(r\sigma_{e}-\sigma_{p})}\int_{0}^{1}\left(f_{r}\cos\left(\frac{k\pi s}{2}\right)\right)'\sin\left(\frac{k\pi s}{2}\right)ds\right] = 0$$
(59)

or, by accounting for $\check{a}_j^T a_j = 1$ and integrating:

$$i\omega_{kj}B' = -\frac{1}{2}i\omega_{kj}c_{j}B + \frac{k\pi n_{j}}{2} \left(2BF_{0k} + \bar{B}F_{rk}e^{i(r\sigma_{e} - \sigma_{p})t}\right)$$
(60)

where the following positions have been introduced:

$$c_{j} := \check{\mathbf{a}}_{j}^{T} \Xi \mathbf{a}_{j}, \qquad n_{j} := \check{\mathbf{a}}_{j}^{T} \mathbf{N} \mathbf{a}_{j},$$

$$F_{rk}(\gamma) := \int_{0}^{l} \left[f_{r}(s;\gamma) \cos\left(\frac{k\pi s}{2l}\right) \right]' \sin\left(\frac{k\pi s}{2l}\right) ds$$
(61)

By letting $B := B_R + iB_I$ and separating real and imaginary parts, Eq. (60) becomes:

$$\begin{pmatrix} \dot{B}_{R} \\ \dot{B}_{I} \end{pmatrix} = \begin{bmatrix} -\frac{c_{j}}{2} + \frac{kn_{j}\pi F_{rkI}}{2\omega_{kj}} & \frac{r\sigma_{e} - \sigma_{p}}{2} + \frac{kn_{j}\pi}{2\omega_{kj}} (2F_{0k} - F_{rkR}) \\ -\frac{r\sigma_{e} - \sigma_{p}}{2} - \frac{kn_{j}\pi}{2\omega_{kj}} (2F_{0k} + F_{rkR}) & -\frac{c_{j}}{2} - \frac{kn_{j}\pi F_{rkI}}{2\omega_{kj}} \end{bmatrix} \begin{pmatrix} B_{R} \\ B_{I} \end{pmatrix}$$
(62)

where F_{rkR} and F_{rkI} are the real and imaginary parts of F_{rk} , respectively; they read:

$$F_{2kR} = -\frac{k\pi^3}{128}q_1^2\cos(2\gamma_e), \qquad F_{2kI} = -\frac{k\pi^3}{128}q_1^2\sin(2\gamma_e), F_{4kR} = 0, \qquad F_{4kI} = 0, F_{6kR} = -\frac{9k\pi^3}{128}q_3^2\cos(3\gamma_e + \gamma_i), \qquad F_{6kI} = -\frac{9k\pi^3}{128}q_3^2\sin(3\gamma_e + \gamma_i),$$
(63)

The eigenvalues of the 2×2 matrix appearing in Eq. (62) decide about the decaying or diverging evolution of the amplitude of perturbation B, from which the stability of the planar motion.





Fig. 3. The case-study: a) front of the building; b) layout of the generic story, which is symmetric with respect to y-axis.

5. Numerical results

a)

The case-study considered here is shown in Fig. 3, and is constituted by a building of n = 9 stories, with interstory height h = 3 m; N = 9 squared and equal columns are present at each story, symmetrically positioned with respect to the a_y -axis but asymmetrically with respect to a_z -axis. The columns have sides of lenght 0.4 m and the material is concrete (Young modulus E = 30 GPa). At each story, a rigid horizontal squared slab represents the floor of the shear-type frame, which has total surface of 8.0×8.0 m². The building slenderness is $\lambda_b = 3.375$ while the column slenderness $\lambda_c = 26.0$, providing $r_{bs} \simeq 0.18$, which is about an upper limit to consider suitable the shear-beam model. The nondimensional parameters which define the system are: $\xi = 0.05$, $\kappa = 4/\pi^2$, $\eta = 21.1$. The distributed load is assumed as uniform along s, and its nondimensional amplitude is p = 0.013 (corresponding to the dimensional value $p_y = 700$ kN/m). Besides the mentioned values, which allow one to perform the in-plane analysis, the values of the parameters required for the stability to out-of-plane disturbances are:

$$\kappa_w = \frac{4}{\pi^2}, \quad \kappa_{w\theta} = 0.004, \quad \kappa_{\theta w} = 2.43, \quad \kappa_{\theta} = 4.50, \\ n_w = 7.03, \quad n_{w\theta} = 0.065, \quad \eta_{\theta w} = 42.2, \quad \eta_{\theta} = 326.32$$
(64)

Morover, $\xi_w = \xi_{\theta} = \xi = 0.05$ is assumed. With the adopted parameters, the eigenvectors are:

$$a_1 = \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0\\0.999\\-0.594 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0\\0.023\\25.449 \end{pmatrix}$$
 (65)

which correspond to $\omega_{11} = 1$, $\omega_{12} = 0.99$, and $\omega_{13} = 3.33$, respectively, indicating a weak but not vanishing coupling between w and θ due to the asymmetry of the generic story with respect to the a_z -axis.

Focusing the attention to in-plane behavior first, equilibrium solutions of Eqs. (43) are sought as functions of σ_e , for a specific value of the load amplitude parameter, $\alpha = 1$, in order to evaluate amplitudes of periodic motions in v(s,t), as given by Eq. (45) (leaving a complete analysis of the nonlinear response for different values of α to future papers). For this purpose, use is made of the software Auto [25], and solutions are shown in Fig. 4 in terms of q_1 and q_3 , where stability is indicated by the solid line and instability by a dashed line: as it is previously stated considering Eq. (43), it is evident how, for all the values of σ_e in the consistent range, always coupled responses ($q_1 \neq 0, q_3 \neq 0$) occur; moreover softening behavior for q_1 occurs, as well as multiple coexisting solutions in the range $\sigma_e \in [0.15, 0.3]$. Furthermore, a loop of the equilibrium paths occurs in the range $\sigma_e \in [0.23, 0.29]$ (see the corresponding windows in Fig. 5), where a further stable solution appears, whose stability is limited by a Hopf bifurcation point at $\sigma_e = 0.252$. From it, a time-periodic solution in q_1, q_3 emanates, as confirmed by numeric integration of Eqs. (43) (see Fig. 6, for $\sigma_e = 0.250$, giving rise to quasiperiodic solution in v), where the amplitude of the limit cycle is shown by the blue region in Fig. 9 for $\sigma_e = 0.249$, Figs. 8 for $\sigma_e = 0.249$, Figs. 9 for $\sigma_e = 0.249$, parametes, in dependence of the initial conditions.

As second step, the stability of the in-plane solution to out-of-plane disturbances is analyzed. Specifically, the absolute value of the complex functions f_r , r = 0, 2, 4, 6 as defined in Eq. (51), is shown in Fig. 10, where it is evident how the larger contributions to the parametric excitation in system (50) come from f_0 and f_2 , whereas those from f_4 and f_6 are almost negligible.

Furthermore, the evolution of the eigenvalues of the Jacobian matrix of system (62) is shown in Fig. 11, where it is evident how, in the region $\sigma_e \in [0.03, 0.1]$ the real part of one of them becomes positive, indicating instability of the in-plane solutions to disturbances. In the same region, the eigenvalues change their nature, from complex conjugate to real and distinct. The instability occurrence is highlighted in Fig. 12, where the region in which the response-frequency function of the in-plane motion becomes unstable to out-of plane disturbances is marked in pink.





Fig. 4. Frequency response functions in q_1 and q_3 . Solid line: stable; dashed line: unstable; HB: Hopf bifurcation.



Fig. 5. Frequency response functions in q_1 and q_3 . Solid line: stable; dashed line: unstable; HB: Hopf bifurcation; blue region: quasi-periodic solutions.



Fig. 6. Periodic solution in q_1, q_3 for $\sigma_e = 0.250$, which produces quasiperiodic solution in v. a) Phase plot; b) power spectrum S_{q_1} for q_1 .



Fig. 7. Periodic solution in q_1, q_3 for $\sigma_e = 0.249$, which produces quasiperiodic solution in v. a) Phase plot; b) power spectrum S_{q_1} for q_1 .



Fig. 8. Quasiperiodic solution in q_1 , q_3 for $\sigma_e = 0.248$, which produces quasiperiodic solution in v. a) Phase plot; b) power spectrum S_{q_1} for q_1 .



Fig. 9. Chaotic solution in q_1, q_3 for $\sigma_e = 0.245$, which produces quasiperiodic solution in v. a) Phase plot; b) power spectrum S_{q_1} for q_1 .







Fig. 12. Frequency response functions in q_1 and q_3 . Solid line: stable; dashed line: unstable; pink region: unstable to out-of-plane disturbances.

6. Conclusions

Nonlinear response and stability of a homogeneous model of tall building is analyzed in the paper. First, recall on the hypotheses under which the homogeneous model of shear-shear-torsional beam is considered as suitable to roughly describe the dynamics of a building is given. Then, a symmetric configuration with respect to a principal axis of the generic story is assumed, so as to focus the attention to the nonlinear in-plane response under external resonant excitation. The Multiple Scale Method is used, considering the internal 1:3 resonance which is a natural occurrence due to the features of the system. Then, stability of the in-plane response to out-of-plane disturbances is analyzed as well, where the relevant variational system, which is linear and parametrically excited by a multi-frequency solicitation, is still tackled via the Multiple Scale Method. A numerical example is proposed, showing the occurrence of multi-modal periodic coexisting solution, which all involve frequency 1 and 3 components. Secondary bifurcations produce quasi-periodic and chaotic solutions as well. Finally, the out-of-plane disturbances are proved to be able to make the in-plane solution unstable in a certain range of frequency.

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Appendix A Expression of the elastic coefficients

The expression of the elastic and geometric coefficients in Eq. (6) is given here for the case of a building where the generic story is constituted by N slender columns, whose centerline is initially located at a positions $\mathbf{x}_i = y_i \mathbf{a}_y + z_i \mathbf{a}_z$, i = 1, ..., N, height h, with rectangular cross-sections of sides b_i , h_i , aligned to \mathbf{a}_y and \mathbf{a}_z , respectively, area $A_i = b_i h_i$, second principal area moments $I_{iy} = b_i h_i^3/12$, $I_{iz} = b_i^3 h_i/12$, torsion inertia moment $J_i = \beta b_i h_i^3$ ($\beta = 0.1406$ for $b_i = h_i$), Young modulus E, transversal elasticity modulus G.



Defining the column axial stiffness $D_i = EA_i/h$, shear stiffness $S_{iy} = 12EI_{iy}/h^3$ and $S_{iz} = 12EI_{iz}/h^3$, and torsion stiffness $C_i = GJ_i/h$, the elastic coefficients in Eqs. (6)-(8) are:

$$D := \sum_{i=1}^{N} D_i, \quad S_y := \sum_{i=1}^{N} S_{iy}, \quad S_z := \sum_{i=1}^{N} S_{iz}, \quad C := \sum_{i=1}^{N} (C_i + S_{iy}y_i^2 + S_{iz}z_i^2)$$
(66)

The coordinates of the extensional and shear center are:

$$y_E := \frac{1}{D} \sum_{i=1}^N D_i y_i, \quad z_E := \frac{1}{D} \sum_{i=1}^N D_i z_i, \quad y_S := \frac{1}{S_y} \sum_{i=1}^N S_{iy} y_i, \quad z_S := \frac{1}{S_z} \sum_{i=1}^N S_{iz} z_i$$
(67)

The inertia radii are:

$$\varrho_{yy}^{2} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} y_{i}^{2}, \quad \varrho_{yz}^{2} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} y_{i} z_{i}, \qquad \varrho_{zz}^{2} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} z_{i}^{2}
\varrho_{yyy}^{3} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} y_{i}^{3}, \quad \varrho_{yyz}^{3} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} y_{i}^{2} z_{i}, \qquad \varrho_{yzz}^{3} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} y_{i} z_{i}^{2}
\varrho_{zzz}^{3} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} z_{i}^{3}, \qquad \varrho_{G}^{2} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} \left(y_{i}^{2} + z_{i}^{2} \right), \quad \varrho_{GG}^{4} \coloneqq \frac{1}{D} \sum_{i=1}^{N} D_{i} \left(y_{i}^{2} + z_{i}^{2} \right)^{2}$$
(68)

