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Research Paper

Stability Analysis of a Damped Nonlinear Wave Equation

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Abstract. The current manuscript is concerned with extracting an analytical approximate periodic solution of a damped cubic nonlinear Klein-Gordon equation. The Riemann-Liouville fractional calculus is utilized to obtain an analytic approximate solution. The Homotopy technique is absorbed in the multiple time-spatial scales. The approved scheme yields a generalization of the Homotopy equation; whereas, two different small parameters are adapted. The first parameter concerns with the temporal perturbation, simultaneously, the second one is accompanied by the spatial one. Therefore, the analytic approximate solution needs the two perturbation expansions. This approach conducts more advantages in handling the classical multiple scales method. Furthermore, the initial conditions are included throughout the multiple scale method to achieve a special solution of the governing equation of motion. The analysis ends up deriving two first-order equations within the extended variables and their actual solution is achieved. The procedure adopted here is very promising and powerful in managing similar numerous nonlinear problems arising in physics and engineering. Furthermore, the linearized stability of the corresponding ordinary Duffing differential equation is analyzed. Additionally, some phase portraits are shown.

Keywords: Klein-Gordon Wave Equation; Fractional Calculus; Homotopy Perturbation Method; Multiple-Scales Method; Stability Analysis; Linearized Stability Method.

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1. Introduction

As well-known, there exists a class of relativistic part of the Schrödinger equation that may be described by the Klein–Gordon equation. Therefore, this equation plays an important role in wide branches of different physical potentials. The study of the identification problems of the system parameters is vague and very vital in the analysis of systems, for instance, see Refs. [1-3]. In the case of the unknown diffusion parameter, they proved the existence of the optimal parameter and deduced the necessary conditions of this parameter. The maximum principles of the optimal control problems, governed by a damped Klein–Gordon equation with state constraints, were examined by Parka and Jeong [4]. Lin and Cui [5] discussed a damped nonlinear Klein-Gordon equation by considering a kernel space. They provided a new technique in solving this equation, simultaneously, to confirm the feasibility and accuracy of the method. Khalid et al. [6] presented a good approach to solve linear/nonlinear Klein-Gordon equations. This algorithm is based on a coupling of the Laplace transforms and perturbation iteration method. They showed that their technique led to a rapidly convergent series. El-Dib [7] introduced the multiple-scale Homotopy technique (He-multiple-scale method) as an outer perturbation of the nonlinear Klein–Gordon equation. Recently, Pang and Yang [8] examined a solution of the initial value problem of a strongly damped Klein-Gordon equation. As established on an adaptation of the concavity method, they considered the coefficients of the dissipative damping terms. D'Abbicco and Ikehata [9] examined the strongly damped Klein-Gordon equation and derived asymptotic profiles of solutions. Recently, El-Dib et al. [10] conducted a new approach to studying the nonlinear azimuthal instability analysis. They transferred the characteristic equation to a Klein-Gordon equation. Through a traveling–wave solution, they examined the stability profile. Additionally, El-Dib and Elgazery [11] applied the properties of the fractional calculus to analyze a damping nonlinear oscillator. As well as, Elgazery [12] introduced a periodic solution for the Newell-Whitehead-Segel model by utilizing the fractional calculus.

Fractional calculus is an old topic. In reality, it has an almost similar history as that of the classical calculus; for instance, see Miller and Ross [13]. The reader can find a comprehensive book on this topic; for instance, see Samko et al. [14]. Recently, in many references cited there, physicists and engineers realized that those differential equations may be formulated along with the fractional derivative. It should be noted that various kinds of real problems are modeled with the aid of fractional calculus. For instance, visco-elastic systems, signal processing, diffusion processes, control processing, fractional stochastic systems, allometry in biology, signal processing, anomalous diffusion, and ecology; for instance, see Refs. [15-17]. In contrast with classical derivatives, there are many kinds of definitions of fractional derivatives. These definitions are generally not equivalent to each other. Li et al. [18] introduced a further study of the vital properties of the Riemann-Liouville derivative. Some important properties of the

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Caputo derivative which have not been discussed, elsewhere, were simultaneously mentioned. Furthermore, they generalized the fractional derivative that defined on the real line of the partial fractional derivatives in higher space dimensions. Valério et al. [19] introduced a comprehensive review of useful established formulae in the field of fractional calculus. Shadab et al. [20] exploited a new and interesting Riemann–Liouville type of fractional derivative operator. El-Dib [21] formulated an analytic approximate solution of a fractional-delayed damping Duffing oscillator.

In reality, the vast phenomena of physics and engineering are modeled via partial differential equations (PDE). Simultaneously, many attractive physically PDEs are arising in different situations. The analytic exact solutions of these equations are rather difficult. The multiple time scales method [22] is utilized in obtaining the stability analysis of these solutions. It is still the most important technique in this topic. Once more, the higher orders yield another amplitude equation or solvability conditions [23]. Finally, one may combine these equations in a single equation; for instance, see Refs. [7, 24-25]. The solution of the latter equation resulted in the stability criteria. Consequently, a uniform valid expansion arises. Many researchers have used the multiple time scales method in their analysis. It is worthwhile to notice that the classical multiple time scales introduced by Nayfeh [22] considered a single small parameter, namely; ε , where $T_n=\varepsilon^n t$, , and $X_n=\varepsilon^n x$, where x and t are the two independent variables. In contrast, our current work includes two small parameters, namely; δ and ρ , where $T_n=\delta^n t$, and $X_n=\rho^n x$, where δ and ρ are the two different small parameters; for instance, see Refs. [26 and 27]. This new approach yields more modifications on the stability configuration. The higher-order multiple scale method were studied by Luongo and Paolone [28] and El-Dib [27]. Ghayesh et al. [29] found a general analytical solution of the nonlinear vibration of the parametric excited continuous system. A delayed epidemic model with nonlinear was examined by Wang and Chen [30]. Moreover, they derived the normal form of the Hopf bifurcation. Finally, the validity of analytic results was shown by their consistency with numerical calculations. Wang et al. [31] applied the tools of the functional analysis to examine the uniqueness and existence of multi-scale fractional stochastic neural network solutions. By constructing a descent Lyapunov functional, the asymptotic stability of the solution of the given problem was examined. Recently, Moatimid [32] examined the motion of a sliding bead on a smooth wire, which is bent in the shape of a vertical parabola. He utilized the multiple time-scales method in his investigation.

The basic concept of the Homotopy perturbation method (HPM) first planned by Ji-Huan He [33]. The major property of the HPM is in its ability and flexibility to examine a wide class in nonlinear differential equations conveniently and accurately. It has more developed and improved by scientists and engineers, for example, a coupled of the Homotopy perturbation method with the Laplace transforms were performed by El-Dib and Moatimid [34]. The HPM with two expanding parameters was suggested by He [35] and El-Dib [27]. The method is effective for some partial nonlinear equations. The two most significant steps in the criteria of the HPM were constructed by He [36] with a suitable initial guess. El-Dib [37] suggested a modified version of the HPM by the multiple scales technique. This new modification works particularly well for the nonlinear oscillators. Ren et al. [38] made a couple of the multiple time scales with the HPM to become a powerful mathematical tool for various nonlinear equations. A novel approach in examining a nonlinear Rayleigh-Taylor instability is conducted by El-Dib et al. [39]. Away from the traditional techniques, they utilized the HPM. Furthermore, along with the expanded frequency analysis, they achieved an analytical periodic solution of the surface deflection.

Moatimid et al. [40] examined an unsteady instability of three horizontal superposed conducting incompressible fluids. Their analysis revealed an Ince's equation. They adapted the He's multiple scales method to analyze the stability analysis. El-Dib [27] introduced a new technique to solve a nonlinear PDE, via the HPM with double expanding parameters. Two Homotopy perturbation expansions, namely the outer and inner perturbations. El-Dib and Mady [41] examined He's-multiple-scale scale to analyze the cubic-quintic Duffing equation arising in the nonlinear instability of two rotating magnetic fluids. Recently, Moatimid [42] investigated a parametric Duffing oscillator. Different methods were utilized to achieve analytic approximate solutions to the problem. The HPM with the fractional multiple scales is utilized to study delayed nonlinear Duffing oscillators [25].

From the aforementioned topics, the Klein-Gordan equation was described by a nonlinear partial differential equation that indicated the behavior of many practical problems arising in engineering, physics, and in many real-world applications. Therefore, the major contribution of the current study is to extend our work as given in Refs. [7. 27]. Consequently, the main purpose is to absorb the temporal-spatial multiple scales into the HPM. Each of the spatial, as well as the temporal multiple scales, are expanded by using two different small Homotopy parameters. To crystallize the presentation of the present paper, the remaining of it is organized as follows: Section 2 is devoted to introducing governing nonlinear Klein-Gordon wave equation along with the boundary as well as the initial conditions. The fractional calculus of the governing equation of motion is depicted in Section 3, additionally, with the amplitude-frequency. The procedure of multiple-scale technology is depicted in Section 4. Section 5 is depicted to display the linearized stability to the corresponding Duffing equation. Furthermore, some phase portraits are plotted. Finally, the concluding remarks are drawn in Section 6.

2. The Mathematical Model

Throughout this section, the technique of the temporal-spatial multiple scales with two different Homotopy small parameters will be applied. The derived solution has a general form away from the dependence on the traveling wave assumption. Furthermore, the usage of the multiple-scales method considers the initial or boundary conditions. This approach provides a new skill to the reader. For this objective, consider the following damped nonlinear Kelin-Gordan model. This model is considered as one of the most interesting wave equations, which arise in relativistic quantum mechanics. Additionally, it has great importance in vast practical applications in physics, engineering, and many other scientific fields; for instance, see Refs. [43, 44]. The current paper considers the following equation:

$$y_{tt}+Py_{xx}+2\eta y_t+2\mu y_x+\omega^2 y=Qy^3; \quad y=y(x,t), \tag{1}$$

where the coefficients P, η, μ, ω^2 and Q are real constants. The parameter η represents the temporal damped coefficient, μ stands for the spatial damped coefficient, ω refers to the natural frequency, and Q stands for the cubic-stiffness parameter. Moreover, this system is subjected to the following collection of the initial and boundary conditions:

$$y(x,0)=\varphi(x),\ \ y_t(x,0)=0,\ \ \text{and}\ \ y(0,t)=u(t),\ \ y_x(0,t)=0. \tag{2}$$

3. The Fractional Damped Klein-Goron Equation

In this section, an attempt will be made to achieve an analytic approximate solution of the damped Klein-Gordon equation in light of the fractional calculus approach. Therefore, consider the following temporary fractional form:

$$D_t^{\alpha+1}y + Py_{xx} + 2\eta D_t^{\alpha}y + 2\mu y_x + \omega^2 y = Qy^3, \quad 0 < \alpha \le 1 \tag{3}$$

The first fractional derivative in eq. (3) represents a generalization of the time-partitioned integer-order derivative in the classical oscillation process. Meanwhile, the second fractional derivative is a generalization of the time-damping term with the positive constant-coefficient 2η.

About the fractional derivative definition, many mathematicians start from different approaches and give different definitions of the fractional derivatives. The rationality and significance of these definitions have been tested in practice. The development of this branch has been widely used in practical problems. So far, there are about four commonly used definitions of fractional derivatives and differentials. These are; the Riemann-Liouville fractional derivatives, Grunwald-Letnikov fractional derivatives, Caputo fractional derivatives, and Miller-Ross Sequential fractional derivatives. In references [13, 45, 46 and 47], various properties, as well as definitions fractional calculus, were provided. Because we aim to obtain a periodic approximate solution, the current article, the Riemann-Liouville fractional derivative and integral will be utilized.

3.1 Homotopy equation with an auxiliary equivalence technique

In this subsection, we are concerned to establish the Homotopy equation by using the auxiliary equivalence technique; for instance, see Refs. [11, 26]. Therefore, eq. (3) can be distinguished as

$$L(y) + R(y) + N(y) = 0, \tag{4}$$

where the operators L, R and N are defined as:

$$L(y) = D_t^{\alpha+1}y + \omega^2 y,\ R(y) = Py_{xx} + 2\eta D_t^{\alpha}y + 2\mu y_x,\ N(y) = -Qy^3. \tag{5}$$

Following similar arguments as given by El-Dib and Elgazery [11], and usage of the auxiliary equivalence technique [11 and 26]. Therefore, one can operate on both sides of eq. (4) by $D_t^2 L^{-1}$, consequently, one gets

$$D_t^2 y + D_t^2 L^{-1}\left(Py_{xx} + 2\eta D_t^{\alpha}y + 2\mu y_x - Qy^3\right) = 0. \tag{6}$$

Introducing a new auxiliary parameter $\Omega^2(x)y$, it follows that the above equation becomes

$$D_t^2 y + \Omega^2 y = \Omega^2 y - D_t^2 L^{-1}\left(Py_{xx} + 2\eta D_t^{\alpha}y + 2\mu y_x - Qy^3\right), \tag{7}$$

where the operator L^{-1} is an integral form of the linear operator L. At this end, one can build the Homotopy equation in the following form:

$$D_t^2 y + \Omega^2 y = \rho\left[\Omega^2 y - D_t^2 L^{-1}\left(Py_{xx} + 2\eta D_t^{\alpha}y + 2\mu y_x - Qy^3\right)\right];\ \rho \in [0,1]. \tag{8}$$

Consider that the solution $y(x,t;\rho)$ may be expanded as

$$y(x,t;\rho) = y_0(x,t) + \rho y_1(x,t) + \rho^2 y_2(x,t) + \dots . \tag{9}$$

Inserting eq. (9) into eq. (10), then set all the identical powers of ρ to zero, one finds

$$\rho^0 : D_t^2 y_0 + \Omega^2 y_0 = 0;\ y_0(x,0) = \varphi(x),\ y_{0t}(x,0) = 0, \tag{10}$$

$$\rho^1 : (D_t^2 + \Omega^2)y_1 = \Omega^2 y_0 - D_t^2 L^{-1}\left(Py_{0xx} + 2\eta D_t^{\alpha}y_0 + 2\mu y_{0x} - Qy_0^3\right);\ y_1(x,0) = 0,\ y_{1t}(x,0) = 0, \tag{11}$$

The exact solution of Eq. (10) has the form

$$y_0(x,t) = \varphi(x)\cos(\Omega t) \tag{12}$$

Inserting the solution that is given in eq. (12) into eq. (11) and then applying the formulas that were approved by El-Dib and Elgazery [11] as follows:

$$D_t^{\alpha}\cos\Omega t = \Omega^{\alpha}\cos\left(\Omega t + \tfrac{1}{2}\pi\alpha\right), \tag{13}$$

$$\left(D^{\alpha+1} + \omega^2\right)^{-1}(\cos\Omega t) = \frac{\left[\omega^2\cos\Omega t + \Omega^{\alpha+1}\sin(\Omega t - \tfrac{1}{2}\pi\alpha)\right]}{\left(\Omega^{2\alpha+2} + \omega^4 - 2\omega^2\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)} \tag{14}$$

Therefore, one finds

$$\begin{aligned}(D_t^2 + \Omega^2)y_1 = {}& \Omega^2\frac{\left(P\varphi'' + 2\mu\varphi' - \frac{3}{4}Q\varphi^3\right)\Omega^{\alpha+1}\cos(\tfrac{1}{2}\pi\alpha) + 2\eta\Omega^{\alpha}\varphi(\Omega^{\alpha+1} - \omega^2\sin(\tfrac{1}{2}\pi\alpha))}{\left(\Omega^{2\alpha+2} + \omega^4 - 2\omega^2\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)}\sin\Omega t + \Omega^2\left[\varphi + \vphantom{\frac{1}{1}}\right.\\ &\left. + \frac{\left(P\varphi'' + 2\mu\varphi' - \frac{3}{4}Q\varphi^3\right)(\omega^2 - \Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)) + 2\eta\Omega^{\alpha}\omega^2\varphi\cos(\tfrac{1}{2}\pi\alpha)}{\left(\Omega^{2\alpha+2} + \omega^4 - 2\omega^2\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)}\right]\cos\Omega t - \tfrac{9}{4}\Omega^2 Q\varphi^3\frac{\left[\omega^2\cos 3\Omega t + (3\Omega)^{\alpha+1}\sin(3\Omega t - \tfrac{1}{2}\pi\alpha)\right]}{\left((3\Omega)^{2\alpha+2} + \omega^4 - 2\omega^2(3\Omega)^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)}.\end{aligned} \tag{15}$$

To avoid the presence of the secular terms, eliminating all terms that arise as the coefficients of $\cos\Omega t$ and $\sin\Omega t$, therefore, one gets

$$\left(P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3\right)\Omega\cos(\tfrac{1}{2}\pi\alpha)+2\eta\varphi(\Omega^{\alpha+1}-\omega^2\sin(\tfrac{1}{2}\pi\alpha))=0. \tag{16}$$

$$\left(\Omega^{2\alpha+2}+\omega^4-2\omega^2\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)\varphi+\left(P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3\right)\left(\omega^2-\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)+2\eta\Omega^{\alpha}\omega^2\varphi\cos(\tfrac{1}{2}\pi\alpha)=0. \tag{17}$$

Following the conditions (16) and (17), the uniform valid solution of eq. (15) becomes

$$y_1(x,t)=\frac{9\Omega^2Q\varphi^3}{32\left((3\Omega)^{2\alpha+2}+\omega^4-2\omega^2(3\Omega)^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)}\begin{bmatrix}\omega^2(\cos 3\Omega t-\cos\Omega t)+(3\Omega)^{\alpha+1}[\sin(3\Omega t-\tfrac{1}{2}\pi\alpha)-\sin(\Omega t-\tfrac{1}{2}\pi\alpha)]\\ -\dfrac{3}{2}\Omega[\sin(\Omega t-\tfrac{1}{2}\pi\alpha)+\sin(\Omega t+\tfrac{1}{2}\pi\alpha)]\end{bmatrix}. \tag{18}$$

Therefore, one can construct higher-order problems easily. To avoid the length of the paper, the solution up to the first-order problem is enough to produce satisfactory results. If the expansion in Eq. (9) is convergent at $\rho=1$, it follows that the approximate solution becomes

$$\begin{aligned}y(x,t)&=\varphi(x)\cos\Omega t\\&+\frac{9\Omega^2Q\varphi^3}{32\left((3\Omega)^{2\alpha+2}+\omega^4-2\omega^2(3\Omega)^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)}\begin{bmatrix}\omega^2(\cos 3\Omega t-\cos\Omega t)+(3\Omega)^{\alpha+1}[\sin(3\Omega t-\tfrac{1}{2}\pi\alpha)-\sin(\Omega t-\tfrac{1}{2}\pi\alpha)]\\ -\dfrac{3}{2}\Omega[\sin(\Omega t-\tfrac{1}{2}\pi\alpha)+\sin(\Omega t+\tfrac{1}{2}\pi\alpha)]\end{bmatrix}.\end{aligned} \tag{19}$$

As a limiting case as $\alpha=1$, the solution of the original damped Klein-Gordon eq. (1) arises in the form

$$y(x,t)=\varphi(x)\cos\Omega t+\frac{9\Omega^2Q\varphi^3}{32(\omega^2-(3\Omega)^2)}[\cos 3\Omega t-\cos\Omega t] \tag{20}$$

3.2 The Amplitude-Frequency Formula

The two solvability conditions that are present in Eqs. (16) and (17) can be used to formulate the frequency equation by combing them. This combination may be cross by eliminating the function $\cos(\tfrac{1}{2}\pi\alpha)$. Therefore, one acquires

$$\left(\Omega^{2\alpha+2}+\omega^4-2\omega^2\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)\varphi\left(P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3\right)+\left(P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3\right)^2\left(\omega^2-\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)\right)+(2\eta\varphi\omega)^2\Omega^{\alpha-1}(\Omega^{\alpha+1}-\omega^2\sin(\tfrac{1}{2}\pi\alpha))=0. \tag{21}$$

As a limiting case as $\alpha\to 1$, leads to $\sin(\tfrac{1}{2}\pi\alpha)\to 1$, consequences,

$$\Omega\left(\omega^2-\Omega^2\right)^2\varphi\left(P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3\right)+\Omega\left(P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3\right)^2\left(\omega^2-\Omega^2\right)+(2\eta\varphi\omega)^2\Omega(\Omega^2-\omega^2)=0. \tag{22}$$

For $\Omega\neq 0$ and $\Omega^2\neq\omega^2$, one finds

$$\Omega^2=\omega^2+\frac{1}{\varphi}\left(P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3\right)-\frac{4\eta^2\varphi\omega^2}{P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3} \tag{23}$$

It should be noted that the stability occurs, when the amplitude-frequency Ω^2 be positive, i.e.

$$\omega^2-\frac{4\eta^2\varphi\omega^2}{P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3}+\frac{1}{\varphi}\left(P\varphi''+2\mu\varphi'-\frac{3}{4}Q\varphi^3\right)>0 \tag{24}$$

This is the stability condition for the classical damping Klein-Gordon eq. (1). It should be noted that the solution (20) and the frequency-amplitude eq. (23) cannot be obtained by applying the classical HPM. As shown from the above analysis, the fractional calculus enables us to find these results. In what follows anther approach can be used to analyze the damping nonlinear damping Klein-Gordon eq. (1).

4. An Alternative Approach and the Generalized Homotopy Equation

Since the objective of the multi-Homotopy methodology; for instance, see El-Dib [27] and Pasha et al. [48], one may distinguish between the following two linear partial differential operators as:

$$L_t y=y_{tt}+\omega^2 y \quad\text{and}\quad L_x y=P(y_{xx}+\sigma^2 y), \tag{25}$$

where the unknown parameter σ^2, stands for the square of an artificial frequency, it has been introduced to obtain an oscillatory spatial solution. It will be determined from the constraining of the periodic solution.
Assuming that there are two small Homotopy parameters ρ and δ, such that the generalized Homotopy equation may be constructed as follows:

$$(D_{tt}+\omega^2)y+\rho\Big[P(D_{xx}+\sigma^2)y+\delta\big(2\eta y_t+2\mu y_x-P\sigma^2 y-Qy^3\big)\Big];\quad \rho\in[0,1],\,\delta\in[0,1], \tag{26}$$

where, D_{tt} and D_{xx} are the total temporal and spatial second-order derivatives, respectively.

In this case, the dependent-function may be considered as $y=y(x,t;\rho;\delta)$. Subsequently, eq. (26) is called the Homotopy equation with the multi-expanded parameters; for instance, see El-Dib [37]. As a limiting case as both $\rho\to 1$ and $\delta\to 1$, eq. (26) turns out to be in the original eq. (1). It is clear that as the parameter $\delta\to 1$, the standard Homotopy equation represents a single expanded parameter ρ. The perturbation along with the parameter ρ is named as the main perturbation (or the outer perturbation). On the other side, the perturbation by using the parameter δ is called the marginal perturbation (or the inner perturbation).
Since the promising primary solution occurs, the parameter of the main perturbation (ρ) should tend to zero, therefore, one finds

$$y_0(x,t)=\lim_{\rho\to 0}\lim_{\delta\to 0}y(x,t;\rho;\delta). \tag{27}$$

The final form of the approximate solution of the original eq. (1) has the following form:

$$y(x,t)=\lim_{\rho\to 1}\lim_{\delta\to 1}y(x,t;\rho;\delta). \tag{28}$$

4.1. Solution throughout a Modified Multiple-Scale Technology

To continue in obtaining the solution of the Homotopy eq. (4) by using the multiple-scale technique, the expansion of the function $y(x,t;\rho;\delta)$ may be written into two power series (the outer and inner perturbations) along with the two Homotopy parameters ρ and δ. In achieving these expansions, one may utilize the methodology of two-temporal scales T_0,T_1 and another two spatial-scales X_0,X_1, as stated throughout the introduction, such that $T_n=\rho^n t$ and $X_n=\delta^n x$, $n=0,1$. For the destination of the outer perturbation, the following expansion is considered:

$$y(x,t;\rho,\delta)=y_0(x,T_0,T_1;\delta)+\rho y_1(x,T_0,T_1;\delta)+\rho^2 y_2(x,T_0,T_1;\delta)+\dots, \tag{29}$$

where $y_n(x,T_0,T_1;\delta)$; $n=0,1,2,\dots$ are unknown functions to be determined later.
Consequently, the first and the second-order of the temporal and the spatial partial derivatives will be transformed into

$$(\partial_t,\partial_x)=(D_{T_0}+\rho D_{T_1}+\dots,D_{X_0}+\delta D_{X_1}+\dots) \tag{30}$$

$$(\partial_{tt},\partial_{xx})=(D^2_{T_0}+2\rho D_{T_0}D_{T_1}+\dots,D^2_{X_0}+2\delta D_{X_0}D_{X_1}+\dots) \tag{31}$$

Furthermore, the initial conditions become

$$y(x,0,T_1;\rho,\delta)=y_0(x,0,T_1;\delta)+\rho y_1(x,0,T_1;\delta)+\rho^2 y_2(x,0,T_1;\delta)+\dots=\varphi(x,T_1;\delta), \tag{32}$$

$$y_t(x,0,T_1;\rho,\delta)=D_0y_0(x,0,T_1;\delta)+\rho\big[D_0y_1(x,0,T_1;\delta)+D_1y_0(x,0,T_1;\delta)\big]+\dots=0. \tag{33}$$

In light of the time-multiple-scales, the transformations that are given in Eqs. (30) and (31), the Homotopy equation (26) becomes

$$(D^2_{T_0}+\omega^2)y+\rho\Big\{p(y_{xx}+\sigma^2)y+(2D_{T_0}D_{T_1}+\dots)y+\delta\big[2\eta(D_{T_0}+\rho D_{T_1}+\dots)y+2\mu y_x-P\sigma^2y-Qy^3\big]\Big\}=0, \tag{34}$$

Replacing the function $y(x,t;\rho,\delta)$ by its expansion as given in eq. (29) into the Homotopy equation that is given in eq. (34), and then equating the identical powers of ρ on both sides, the resulting zero and first-order problems are given as follows:

$$(D^2_{T_0}+\omega^2)y_0(x,T_0,T_1;\delta)=0;\quad y_0(x,0,T_1;\delta)=\psi(x,T_1;\delta)\ \&\ D_{T_0}y_0(x,T_1;\delta)=0, \tag{35}$$

$$\begin{gathered}(D^2_{T_0}+\omega^2)y_1(x,T_0,T_1;\delta)=-P(D_{xx}+\sigma^2)y_0-2D_{T_0}D_{T_1}y_0-\delta\big[2\eta D_{T_0}y_0+2\mu D_xy_0-P\sigma^2y_0-Qy_0^3\big];\\ y_1(x,0,T_1;\delta)=0\ \&\ D_{T_0}y_1(x,0,T_1;\delta)+D_{T_1}y_0(x,0,T_1;\delta)=0.\end{gathered} \tag{36}$$

The special solution of the partial differential eq. (35) has the following form:

$$y_0(x,T_0,T_1;\delta)=\psi(x,T_1;\delta)\cos\omega T_0. \tag{37}$$

To obtain a uniform solution of the first-order in ρ, all terms that producing secular terms must be removed. Therefore, all terms that contain the functions $\cos\omega T_0$ or $\sin\omega T_0$ should be canceled. The elimination of all the coefficients of $\sin\omega T_0$ leads to the following linear temporal partial first-order amplitude equation:

$$D_{T_1}\psi+\delta\eta\psi=0. \tag{38}$$

The removing of all the coefficients of $\cos\omega T_0$ giving the following spatial partial second-order amplitude equation:

$$\left(D_{xx}+\sigma^2\right)\psi+\delta\left(2\frac{\mu}{P}D_x-\sigma^2\right)\psi-\frac{3\delta}{4P}Q\psi^3=0. \tag{39}$$

According to the cancelation of these secular terms, the final periodic solution of the function $y_1(x,T_0,T_1;\delta)$ may be written as

$$y_1(x,T_0,T_1;\delta)=\frac{1}{32\omega^2}\delta\left[-Q\psi^3(\cos 3\omega T_0-\cos\omega T_0)\right]-\frac{1}{\omega}D_{T_1}\psi\sin\omega T_0. \tag{40}$$

Employing Eqs. (38) and (40), yields

$$y_1(x,T_0,T_1;\delta)=-\frac{1}{32\omega^2}\delta Q\psi^3(\cos 3\omega T_0-\cos\omega T_0)+\frac{1}{\omega}\delta\eta\psi\sin\omega T_0. \tag{41}$$

It should be noted that the solvability condition as given in eq. (38) is a linear first-order equation with real coefficients. On the other hand, the solvability condition is given in eq. (39) represents a cubic nonlinear spatial second-order equation. For the specialization of a single iteration process, one may combine Eqs. (37), (41) and the expansion (29) and letting $\rho\to 1$, one gets

$$y(x,t;\delta)=\psi\cos\omega t+\frac{1}{\omega}\delta\eta\psi\sin\omega t-\frac{1}{32\omega^2}\delta Q\psi^3(\cos 3\omega t-\cos\omega t);\ \psi=\psi(x,t;\delta). \tag{42}$$

One must remember that $\psi(x,0)=\varphi(x)$ and $\psi(0,t)=u(t)$.

To obtain the contribution of the function ψ, one needs to solve eq. (39). Indeed, this equation results in a spatial second-order nonlinear Homotopy equation with damping Duffing-type. Therefore, the perturbation method is urgent. Consequently, the function $\psi(x,t;\delta)$ may be expanded, because of the spatial multiple scales as

$$\psi(x,t;\delta)=\psi_0(X_0,X_1,t)+\delta\psi_1(X_0,X_1,t)+\delta^2\psi_2(X_0,X_1,t)+\dots \tag{43}$$

By employing the expansion of the first and the second spatial derivatives that are given in Eqs. (30, 31 and 43) into eq. (39), then eliminate the identical coefficients of all powers of δ, one finds

$$(D_{X_0}^2+\sigma^2)\psi_0=0\ ;\ \text{given}\ \ \psi_0(0,X_1,t)=U(X_1,t),\ D_{X_0}\psi_0(0,X_1,t)=0 \tag{44}$$

$$(D_{X_0}^2+\sigma^2)\psi_1=-2\left(D_{X_1}+\frac{\mu}{P}\right)D_{X_0}\psi_0+\sigma^2\psi_0+\frac{3}{4P}Q\psi_0^3\ ;\ \text{given}\ \ \psi_1(0,X_1,t)=0,\left(D_{X_0}\psi_1(0,X_1,t)+D_{X_1}\psi_0(0,X_1,t)\right)=0. \tag{45}$$

Equation (44) is then having a solution in the following form:

$$\psi_0(X_0,X_1,t)=U(X_1,t)\cos\sigma X_0. \tag{46}$$

Substituting from Eq. (46) into eq. (45), once more, removing the source of the secular terms, one obtains

$$\sigma^2=-\frac{9Q}{16P}U^2(X_1,t), \tag{47}$$

$$D_{X_1}U+\frac{\mu}{P}U=0. \tag{48}$$

The cancelation of the secular terms from eq. (45), leads to acquiring the final uniform solution

$$\psi_1=-\frac{3}{16\times 8}\frac{Q}{P\sigma^2}U^3(\cos 3\sigma X_0-\cos\sigma X_0)-\frac{1}{\sigma}D_{X_1}U\sin\sigma X_0. \tag{49}$$

Remove $D_{X_1}U$ and σ^2 from eq. (49) with the aid of eq. (47), one finds

$$\psi_1=\frac{1}{24}U(\cos 3\sigma X_0-\cos\sigma X_0)+\frac{\mu}{\sigma P}U\sin\sigma X_0. \tag{50}$$

Because of a single iteration process is only needed, the function $\psi(x,t)$ has the following form:

$$\psi(x,t)=U\cos\sigma x+\frac{\mu}{\sigma P}U\sin\sigma x+\frac{1}{24}U(\cos 3\sigma x-\cos\sigma x);\quad U=U(x,t). \tag{51}$$

At this end, Eqs. (38) and (48) having the form

$$\psi_t+\eta\psi=0, \tag{52}$$

$$U_x+\frac{\mu}{P}U=0. \tag{53}$$

It is worthwhile to notice that the function $u(t)=\lim\limits_{x\to 0}U(x,t)$.

Furthermore, Eqs. (52) and (53) having exact solutions in the following forms:

$$\psi(x,t)=\varphi(x)e^{-\eta t}, \tag{54}$$

$$U(x,t)=u(t)e^{-\mu x/P}. \tag{55}$$

Therefore, the oscillatory solutions require

$$PQ < 0,\ \eta > 0,\ P\mu > 0 \tag{56}$$

Similar criteria have previously been shown by El-Dib [7], for another Klein-Gordon equation without any damping terms. Employing Eqs. (54) and (55) into Eq. (51), again return to eq. (42), one finds

$$\begin{aligned} y(x,t) = \lim_{\rho\to 1}\lim_{\delta\to 1} y(x,t;\rho;\delta) = u(t)e^{-\frac{\mu}{P}x}\left(\cos\omega t + \frac{\eta}{\omega}\sin\omega t\right)\left[\cos\sigma x + \frac{\mu}{\sigma P}\sin\sigma x + \frac{1}{24}(\cos 3\sigma x - \cos\sigma x)\right] \\ - \frac{Q}{32\omega^2}u^3(t)e^{-\frac{3\mu}{P}x}(\cos 3\omega t - \cos\omega t)\left[\cos\sigma x + \frac{\mu}{\sigma P}\sin\sigma x + \frac{1}{24}(\cos 3\sigma x - \cos\sigma x)\right]^3, \end{aligned} \tag{57}$$

where the final form of the argument σ is given by

$$\sigma^2 = -\frac{9Q}{16P}e^{-2\frac{\mu}{P}x}u^2(t). \tag{58}$$

As previously seen, eq. (57) gives an analytical approximate solution of the governing Klein-Gordon equation as given in eq. (1). The solution is very complicated. The complexity of the solution backs of the argument of the trigonometry functions that contain a temporal-dependent as well as the spatial one. Because of the stability criteria is simple. Therefore, one thinks that no need for numerical calculations.

5. Linearized Stability

The objective here is to analyze the linearized stability near the equilibrium points. Therefore, one returns to the original governing equation as given in eq. (1). For this purpose, to transform the partial differential equation into an ordinary one, consider the following transformation:

$$\theta = 2\mu x + 2P\eta t \tag{59}$$

In light of this transformation, the governing equation is given in eq. (1), will be transformed to the following ordinary second-order differential equation:

$$\frac{d^2y}{d\theta^2} + \frac{1}{P}\frac{dy}{d\theta} + \frac{\omega^2}{4P\left(P\eta^2 + \mu^2\right)}y - \frac{Q}{4P\left(P\eta^2 + \mu^2\right)}y^3 = 0 \tag{60}$$

Assuming the transformations: $y' = z$, it follows that Eq. (60) is converted to the following system:

$$y' = f(y,z)\ , \qquad z' = h(y,z) \tag{61}$$

where the prime denotes that the differentiation in respect to the independent parameter θ and

$$f(y,z) = z\ ,\ h(y,z) = -\frac{1}{P}z - \frac{\omega^2}{4P\left(P\eta^2 + \mu^2\right)}y + \frac{Q}{4P\left(P\eta^2 + \mu^2\right)}y^3. \tag{62}$$

The fixed points occur at the points (y_0,z_0) , where

$$f(y_0,z_0) = 0 \quad \text{and} \quad h(y_0,z_0) = 0 \tag{63}$$

It follows that

$$z_0 = 0\ , \tag{64}$$

and

$$-\omega^2 y_0 + Q\ y_0^3 = 0 \tag{65}$$

Therefore, there are three fixed points as follows:

$$(0,0)\ ,\ (\frac{\omega}{\sqrt{Q}},0) \quad \text{and} \quad (-\frac{\omega}{\sqrt{Q}},0) \tag{66}$$

Provided that the following condition must be, simultaneously, hold:

$$Q > 0 \tag{67}$$

Now, the functions $f(y,z)$ and $h(y,z)$ will be expanded, by using the Taylor expansion, considering only the linear terms, around the previous fixed points.
One finds the following Jacobian matrix:

$$J = \begin{pmatrix} 0 & 1 \\ \frac{1}{4P(P\eta^2 + \mu^2)}(-\omega^2 + 3Q\,y_0^2) & -\frac{1}{P} \end{pmatrix} \tag{68}$$

At the equilibrium point, the Jacobian determinant becomes

Table 1. Roots of the eigenvalues and the corresponding stability/instability.

	Roots of the Eigenvalues	Stability/Instability
1	Real, distinct, and negative	Stable node
2	Real, distinct, and positive	Unstable node
3	Real, distinct, and of different signs	Saddle point
4	Real, equal, and negative	Stable node
5	Real, equal, and positive	Unstable node
6	Roots are pure imaginary	Stable center
7	Roots are complex conjugate, with negative real part	Stable focus
8	Roots are complex conjugate, with positive real part	Unstable focus

Table 2. Different types of the eigenvalues and the corresponding stability/instability.

	Sample chosen system	Fixed point	Roots of the Eigenvalues	Stability/Instability
1	$P=0.5, \mu=0, \eta=1, \omega=1, Q=1$	$(0,0)$	Real, equal, and negative $\Lambda_{1,2}=-1.$	Stable node See Fig. 1.
2	$P=-0.5, \mu=0, \eta=1, \omega=1, Q=1$	$(0,0)$	Real, equal, and positive $\Lambda_{1,2}=1.$	Unstable node See Fig. 2.
3	$P=1, \mu=0.1, \eta=1, \omega=3, Q=0.1$	$(\pm 9.48683, 0)$	Real, distinct and of different signs $\Lambda_1=-2.67, \Lambda_2=1.67.$	Saddle point See Fig. 3.
4	$P=-10, \mu=1, \eta=0.2, \omega=1, Q=1$	$(\pm 1, 0)$	Roots are complex conjugate, with positive real part $\Lambda_{1,2}=0.5\pm 0.218i$	Unstable focus See Fig. 4.

$$J=\begin{vmatrix} -\Lambda & 1 \\ \dfrac{1}{4P(P\eta^2+\mu^2)}(-\omega^2+3Q\,y_0^2) & -(\dfrac{1}{P}+\Lambda) \end{vmatrix} \tag{69}$$

The above determinant has the following eigenvalues:

$$\Lambda_{1,2}=\frac{1}{2P}\left(-1\pm\sqrt{\frac{\mu^2+P(\eta^2-\omega^2)+3PQ\,y_0^2}{(P\eta^2+\mu^2)}}\right) \tag{70}$$

Typically, if all eigenvalues of the Jacobian have negative real parts, it follows that the equilibrium point is stable. Otherwise, the equilibrium point becomes unstable. The different kinds of stability/instability depend mainly on the nature value of the eigenvalues that are given in **Table 1**. It is more convenient to consider a set of chosen sample system to indicate the stability/instability picture in light of the equilibrium points, consequently, the nature of the eigenvalues. This procedure may be done in **Table 2**.

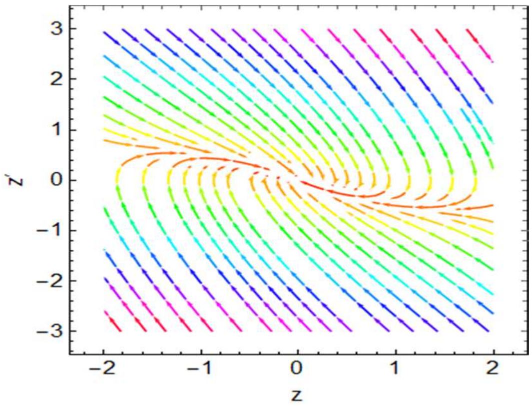

Fig. 1. Depicts the phase portrait for a stable node in a system having the particulars: $P=0.5$, $\mu=0.0$, $\eta=1.0$, $\omega=1.0$ and $Q=1.0$

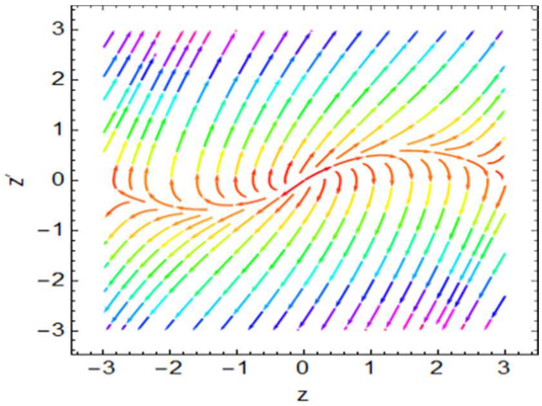

Fig. 2. Displays the phase portrait for an unstable node in a system having the particulars: $P=-0.5$, $\mu=0.0$, $\eta=1.0$, $\omega=1.0$ and $Q=1.0$

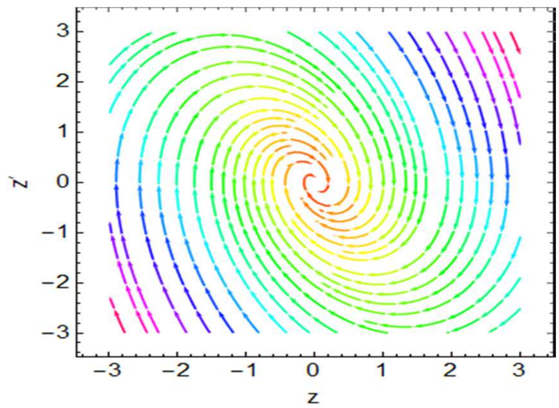

Fig. 3. Plots the phase portrait for a saddle node in a system having the particulars: $P=1.0$, $\mu=0.1$, $\eta=1.0$, $\omega=3.0$ and $Q=0.1$

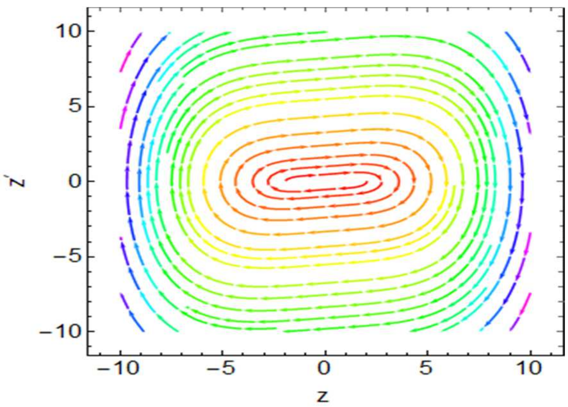

Fig. 4. Graphs the phase portrait for an unstable focus in a system having the particulars: $P=-10.0$, $\mu=1.0$, $\eta=0.2$, $\omega=1.0$ and $Q=1.0$

6. Concluding Remarks

Because of the potential importance of the analytic approximate periodic solution of the Klein-Gordon equation with wide applications in several areas, the sciences physics, and engineering examined this problem. As previously seen throughout the introduction, a set of previous works has been done. The main objective of the different authors, who were interested in solving nonlinear differential equations, is to achieve an analytic approximate solution along with numerical one. Therefore, an attempt to examine such an equation is made. The fractional calculus along with the Riemann-Liouville derivative is performed. Consequently, this paper provides adsorption of the temporal-spatial multiple-scales into the HPM. A novel approach to construct a Homotopy equation with double series, having two increasing parameters, is included. The new formulation is usually recommended by coupling with the temporal and, also, the spatial multiple scales methodology. The cancellation of the secular terms of the first-order perturbation ends in getting two solvability conditions. One of them is represented by a first-order equation that deals with the slow time and its exact solution has been obtained. The second solvability condition has been given in a cubic nonlinear damping Duffing equation in the unknown function of the spatial variable. Consequently, another perturbation is needed. The spatial-multiple scales are applied to solve this equation. Finally, at a single iteration process, the approximate solution is accomplished. The stability conditions have been derived. The current approach can be extended to treat alternative damped nonlinear issues. Subsequently, the present paper may be utilized as a paradigm for alternative applications in the damped nonlinear partial differential equation.

As a conclusion, the following outcomes may be drawn as follows:

- A fractional approximate solution is given in eq. (18).
- The modified Homotopy equation along with the two-small parameters are given in eq. (25).
- The temporal solvability equation is conducted in eq. (36); whereas, the spatial one is given in eq. (47).
- A first iteration process is utilized and, also, the stability criteria are conducted.
- The stability criteria are reported in eq. (55).
- The analytic approximate solution up to the first order is derived in eq. (56).
- The linearized stability of the corresponding Duffing equation is analyzed.
- The eigenvalues corresponding to the equilibrium points are given in eq. (69).
- Some phase portraits near the equilibrium points are plotted.

Author Contributions

The first author proposed and develops the mathematical modeling of the problem and examined the theory validation. The second author introduced the linearized stability. Finally, the third author analyzed the fractional damped Klein-Goron equation. The manuscript was written throughout the contribution of all authors. All authors discussed the outcomes, reviewed, and approved the final version of the manuscript.

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Conflict of Interest

The authors declared that there are no competing interests regarding the publication of the current paper.

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