J. Appl. Comput. Mech., 6(SI) (2020) 1293-1300
DOI: 10.22055/JACM.2020.33778.2285

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

# A Periodic Solution of the Newell-Whitehead-Segel (NWS) Wave Equation via Fractional Calculus

Nasser S. Elgazery

Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt

Received May 30 2020; Revised July 26 2020 ; Accepted for publication July 30 220.
Corresponding author: N.S. Elgazery (nasser522000@gmail.com; nasersaleh@edu.asu.edu.eg)


**Abstract.** The Newell-Whitehead-Segel (NWS) equation is one of the most significant amplitude equations with a wider practical applications in engineering and applied physics. It describes several line patterns; for instance, see lines from seashells and ripples in the sand. In addition, it has several applications in mathematical, chemical, and mechanical physics, as well as bio-engineering and fluid mechanics. Therefore, the current research is concerned with obtaining an approximate periodic solution of a nonlinear dynamical NWS wave model at three different powers. The fractional calculus via the Riemann-Liouville is adopted to calculate an analytical periodic approximate solution. The analysis aims to transform the original partial differential equation into a nonlinear damping Duffing oscillator. Then, the latter equation has been solved by utilizing a modified Homotopy perturbation method (HPM). The obtained results revealed that the present technique is a powerful, promising, and effective one to analyze a class of damping nonlinear equations that appears in physical and engineering situations.

**Keywords**: NWS wave equation; Analytic periodic solution; Nonlinear damping fractional Duffing oscillator; Riemann-Liouville fractional calculus; A modified Homotopy perturbation method.

## 1. Introduction

In wider fields of science, the non-linear partial differential equations have been encountered. This kind of equations had become a hot topic of research. They played a vital role in understanding the physical phenomena in depth especially in solid state physics, fluid mechanics, capillary–gravity waves, plasma waves, as well as chemical physics; for instance, see Polyanin and Zaitsev [1]. The following model formulated in eq. (1) could be considered as One of the most intriguing nonlinear dynamical wave equations, which arises in plasma physics, mathematical biology, and quantum mechanics:

$$\frac{\partial \upsilon(x,t)}{\partial t} = m\frac{\partial^2 \upsilon(x,t)}{\partial x^2} + \kappa \upsilon(x,t) - \beta\big(\upsilon(x,t)\big)^n , \tag{1}$$

Eq. (1) refers to the simplest one spatial dimensional nonlinear reaction–diffusion equation in a plane geometry. Herein, $\kappa$ and $\beta$ are non zero real constants, and $m$ is a positive integer. $t$ and $x$ are the temporal and spatial variables, respectively, where $t \ge 0$ , $x \in \mathbb{R}$ and $\kappa\upsilon - \beta\upsilon^n$ is the nonlinear reaction (source)term. As well as, $\upsilon(x,t)$ may be considered as the flow velocity of a fluid in an infinitely long pipe with a smaller diameter or as the distribution of temperature in an infinitely long and thin bar. The theoretical model of the physical phenomenon is sketched in Fig. 1.

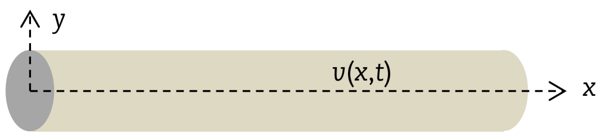


**Fig. 1.** The flow velocity of a fluid in an infinitely long pipe with a small diameter.

The first case is the Allen-Cahn (A-C) equation, which occurs when $n$ and $\kappa = -\beta = 1$ . This problem had various applications in quantum mechanics, biology and plasma physics; for instance, see Hariharan [2]. The second case is the Fishers equation, when $n = 2$ , and $\kappa = -\beta$ . The last one appeared in various studies in describing the Rayleigh-Benard convection of binary fluid mixtures. Therefore, it was called the NWS equation when power took a commonly used value, i.e. $n = 3$ . Herein, in addition to the commonly used value of the power, $n$ , the NWS model with other values such as negative integer and fraction power i.e., $n = -3$ and $1/3$ will be studied.

On the topic of the mathematical physics, there are many techniques to analyze the nonlinear dynamical wave equations.

Newell and Whitehead [3] and Segel [4] studied the nonlinear partial differential parabolic equation. During the past decades, a lot of researchers have been utilized distinct methods in solving the NWS equation. Zellal and Belghaba [5] as well as Nourazar et al. [6] utilized the HPM to get some analytical solutions for this model. Additionally, Jassim [7] used the Laplace transforms to obtain solutions for this equation. Valls [8] proposed an algebraic traveling wave for a description of the generalized NWS equation. Recently, Korkmaz [9] used an expansion method, based on the sine-Gordon equation, to achieve a complex wave solution for the same equation that describes a mathematical biology model. Furthermore, the readers can see other systems in a quantum simulator which based on the sine-Gordon model; for instance, see Refs. [10 and 11]. Moreover, various types of iterative method were utilized to obtain some approximate solutions of the NWS equation. Prakash and Kumar [12] and Akinlabi and Edeki [13] used the He's variational and perturbation iteration transform methods to accomplish solutions of the NWS model. Patade and Bhalekar [14] utilized new iterative method to calculate an analytical periodic approximate solution of the NWS model. Moreover, via the Adomian decomposition method (ADM), some different studies have been applied to the same objective. Aswhad and Jaddoa [15] as well as Prakash and Verma [16] studied the Adomian decomposition method to calculate approximate solutions for the NWS equation. Mahgoub and Sedeeg [17] investigated the combined form of the Adomian decomposition and Elzaki transform methods to solve the NWS equation. In another study, a composite form of the ADM and Laplace transforms was utilized for the same purpose as described by Pue-on [18]. Saravanan and Magesh [19] introduced a comparative study between both of the ADM and reduced differential transform methods for the present model. Recently, Nadeem et al. [20] applied a couple of the He's polynomials and variational iteration technique to achieve an analytical solution as well as an exponential finite difference was applied to find a numerical solution to this equation as shown by Hilal et al. [21]. Additionally, Almousa et al. [22] and Ahmad et al. [23] obtained analytic approximate solutions for this equation via the Mahgoub Adomian decomposition method and modified variations iterative methods, respectively. Furthermore, Saha and Das [24] presented an analytical and a computational approach for solving the perturbed/unperturbed generalized NWS equations.

On the other hand, there are some important articles that were prepared to examine the fractional NWS equation. For instance; Edeki et al. [25] investigated the combined between the He's polynomials and fractional complex transform method. Saadeh et al. [26] applied the fractional residual power series algorithm to calculate approximation solutions to this equation. Moreover, Lie symmetry analysis had been applied to compute an analytical solution of the same equation by Saberi et al. [27]. Additionally, an asymptotic-sequentially solution, based on the Caputo fractional calculus, has been formulated by Ali et al. [28]. Furthermore, a numerical approximation of the fractional NWS equation has been described by Kumar and Sharma [29]. Recently, Prakash et al. [30] proposed a fractional variational iteration approach to solve the same time-fractional equation. In reality, the fractional calculus is considered as an old topic, it was similar as the classical one; for instance, see Miller and Ross [3١]. Since the existence of the fractional calculus, great and significant approached has been developed. Therefore, on the fractional calculus topic, there was a lot of comprehensive books as well as research articles cited therein for example; for instance, see Samko et al. [32]. Also, various engineers, mathematicians, and physicists realized that ordinary/partial differential equations can be reformulated via the fractional derivative. For this reason, the reader should be noted that many real problems were modeled via the fractional calculus. For example, diffusion processes, visco-elastic and fractional stochastic systems, control and signal processing, ecology, and geometry in biology; for instance, see Refs. [33–35]. It should be noted that, there were various fractional derivative definitions. In general, these various definitions were not equivalent. The Riemann-Liouville derivative fractional calculus was considered as a one of the mostly used fractional derivatives; for instance, see Li et al. [36]. A comprehensive review of the various useful established formulas in the topic of the fractional calculus has been presented by Valério et al. [37]. Recently, Shadab et al. [38] exploited an interesting and new Riemann–Liouville formula of a fractional derivative operator. In addition, El-Dib [39] introduced an analytic approximate solution as well as the stability condition of a fractional-delayed damping Duffing equation based on the combination of Homotopy perturbation method with the Riemann-Liouville approach. Furthermore, a system of 3D forced-damped has been analyzed by Llibre et al. [40].

The fractional damped Duffing's equation has been investigated by Padovan and Sawicki [41]. They studied the effect of the damping term with a fractional order of the amplitude-frequency response. The influence of the optimization harmonic balance approach was utilized by Haitao Liao [42] to analyze of two different fractional order derivatives of the nonlinear Duffing equation. El-Dib [43] demonstrated a combination of both of the multiple scales and the parameter expansion in a stability behavior of the fractional derivative to analyze a cubic delayed Duffing oscillator. Shen et al. [44] used the averaging method to investigate the harmonic resonance of the Duffing oscillator fractional equation (two kinds of derivatives orders). Jiménez et al. [45] studied a fractional Duffing equation in the presence of both harmonic and non-harmonic external perturbations, at which the latter one arising in the geometrical resonances. The Homotopy analysis approach was utilized to derive an analytic solution to nonlinear Duffing Oscillator with fractional derivatives by Ejikeme et al. [46]. Recently, El-Dib and Elgazery [47] obtained a periodic solution and stability analysis of the time-fractional damping Duffing equation by using a modified HPM. Furthermore, Shen and El-Dib [48] derived a modification to the HPM to study the fractional sine-Gordon equation [48]. Furthermore, separable functional solutions for a nonlinear telegraph type and Klein–Gordon equations based on symmetry reductions have been introduced by Zhurov and Polyanin [49].

Based on the aforementioned topics, the NWS equation is described by a temporal-spatial nonlinear partial differential equation. It indicated the behavior of a lot of practical problems arising in physics, engineering in various real-world applications. Therefore, the major objective of the current study is to acquire a periodic solution as well as the stability analysis for the NWS model. The present method is mainly based on transforming the original partial differential equation to a nonlinear damping Duffing oscillator. An analytical periodic approximate solution of the latter equation has been formulated via utilizing the Riemann-Liouville fractional calculus and then using a modified HPM. To crystallize the presentation, it is organized as follows: Section 2 is devoted to introducing the temporal-spatial nonlinear NWS wave model. The fractional calculus and the procedure of the present equation are depicted in Section 3. Section 4 is devoted to represent two analytical periodic approximate solutions of fraction and negative integer power, temporal-spatial nonlinear NWS wave models, respectively. Finally, the concluding remarks are drawn in Section 5.

## 2. The Governing Model

Consider the following Newell-Whitehead-Segel (NWS) equation that results from eq. (1), in case of the following parameters:

$$\mu = m^{-1}, \sigma^2 = \kappa m^{-1}, \text{ and } \ Q = -\beta m^{-1}$$

Eq. (1) is then reduced to

$$\mu \frac{\partial \upsilon}{\partial t} = \frac{\partial^2 \upsilon}{\partial x^2} + \sigma^2 \upsilon + Q\upsilon^n, \tag{2}$$

where $\upsilon(x,t)$ is an unknown function that to be determined.
Assuming a spatial variable $\zeta$ such as

$$\upsilon(x,t)=\upsilon(\zeta);\zeta=x-\mu t. \tag{3}$$

It follows that

$$\frac{\partial}{\partial x}\equiv\frac{\partial}{\partial\zeta},\frac{\partial^2}{\partial x^2}\equiv\frac{\partial^2}{\partial\zeta^2} \quad \text{and} \quad \frac{\partial}{\partial t}\equiv-\mu\frac{\partial}{\partial\zeta} \tag{4}$$

Substituting from eq. (3) into the eq. (2), then using eq. (4), one gets

$$\upsilon''(\zeta)+\mu\upsilon'(\zeta)+\sigma^2\upsilon(\zeta)+Q\upsilon^n(\zeta)=0, \tag{5}$$

herein, the prime denotes the differentiation with respect to the independent variable $\zeta$.
The resulting eq. (5) represents a damping Duffing equation with the nonlinear coefficient, $\mu$ and $Q$.
It is worthy to notice that most of the perturbation methods failed to analyze the nonlinear damping Duffing eq. (5); even the straight forward expansion. In the present proposal, one shows throughout the fraction analysis technique, by applying the HPM, that the shortcomings in periodic solution will overcome. Therefore, the fractional analysis approach will overcome this difficulty without any need to apply the currently used methods in the literatures; such as multiple scales or the time-delay [50 and 51].

## 3. The Fractional Derivative

In the development of using the fractional derivative of various practical problems, especially in engineering and physical problems, one should be noticed that there are several kinds of the fractional derivatives definitions. On the other hand, there are approximately four commonly fractional derivatives are prepared by the Riemann-Liouville, Caputo, Grünwald-Letnikov, and Miller-Ross. The various definitions and properties of the fractional calculus were appeared in Refs. [31-32, 37, 52-54]. In addition, the most popular ones being the Riemann-Liouville fractional derivative that plays an important role mainly in the development of the theory of the fractional derivatives. However, the well-known Caputo fractional derivative is less involved since the associated integral operator manipulates the derivatives of the primitive function under the integral symbol; for instance, see [55]. Comparing the Caputo derivative definition with the Riemann-Liouville one, functions which are derivable in the Caputo sense are much fewer than those which are derivable in the Riemann-Liouville sense [56]. Therefore, a lot of mathematical scientists, widely utilized the fractional derivative, depending on the definition of Riemann-Liouville.
Consequently, in the present proposal, the most famous fractional integral/derivative definitions that come from Riemann-Liouville will be used.
The Riemann-Liouville time-fractional derivative of the function in order $0<\alpha\le 1$ is defined as follows:

$$D_a^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t\frac{f(\gamma)}{(t-\gamma)^\alpha}d\gamma; \quad 0<\alpha\le 1;\ t>a \tag{6}$$

Considering the fact that the Riemann-Liouville fractional derivative of $f(t)=e^{i\omega t}, t\in\mathbb{R}, a\to-\infty$; for instance, see Ortigueira et al.[57], one gets

$$I_{-\infty}^\alpha e^{i\omega t}=(i\omega)^{-\alpha}e^{i\omega t} \ \text{ and } \ D_{-\infty}^\alpha e^{i\omega t}=(i\omega)^\alpha e^{i\omega t}, \alpha\in\mathbb{R}. \tag{7}$$

Based on the definition (7), one finds

$$D_{-\infty}^\alpha\cos\omega t=\omega^\alpha\cos\left(\omega t+\tfrac{1}{2}\pi\alpha\right) \ \text{ and } \ D_{-\infty}^\alpha\sin\omega t=\omega^\alpha\sin\left(\omega t+\tfrac{1}{2}\pi\alpha\right). \tag{8}$$

an additional $f(t)$ formula may formulate as follows; for instance, see El-Dib and Elgazery [47]:

$$\left(D^{\alpha+1}+\omega_0^2\right)^{-1}(\cos\omega t)=\frac{[\omega^{\alpha+1}\cos(\frac{1}{2}\pi\alpha)]\sin\omega t+[\omega_0^2-\omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)]\cos\omega t}{\left(\omega^{2\alpha+2}+\omega_0^4-2\omega_0^2\omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)}, \tag{9}$$

and

$$\left(D^{\alpha+1}+\omega_0^2\right)^{-1}(\sin\omega t)=\frac{[\omega_0^2-\omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)]\sin\omega t-[\omega^{\alpha+1}\cos(\frac{1}{2}\pi\alpha)]\cos\omega t}{\left(\omega^{2\alpha+2}+\omega_0^4-2\omega_0^2\omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)}. \tag{10}$$

### 3.1 The Outline Solution of the Fractional Damping Duffing Oscillator

For the purpose of the analytical technique, one reformulates the original eq. (5) at $(n=3)$ to derive a periodic analytical solution of the nonlinear fractional order damping Duffing equation as follows:

$$D_\zeta^{\alpha+1}\upsilon+\mu D_\zeta\upsilon+\sigma^2\upsilon+Q\upsilon^n=0; \quad 0<\alpha\le 1. \tag{11}$$

In the final results, the limiting cases, as $\alpha$ tends to $1$, the final solution of the original eq. (5) will be calculated.
Moreover, here it is necessary to point out that the exact solution of the generalized NWS model with power law nonlinearity and time dependent coefficients is obtained by different methods; in [58] using tanh function method, in [6] by using the Homotopy perturbation technique. Additionally, Nadeem et al.[20] used a variational iteration approach with He's polynomials. The existence and uniqueness of solutions of parabolic equations are discussed by many authors. They used different methods; such as fixed

point theory (using contraction mapping in complete normed spaces), Fourier transform technique and potential theory (transforming the partial differential equation onto integral equation with singular kernel, see Refs. [59 and 60]. Furthermore, the existence and stability of fractional parabolic partial differential equations (as the fractional model (11)) are recently discussed even for fractional integro-partial differential equations with nonlocal condition and nonlinear fractional order partial differential equations with delay, see Refs. [61 and 62].

To construct the modified Homotopy equation of the nonlinear fractional Duffing equation at $n=3$. Firstly, one operates on both sides of eq. (11) by the operator $D_\zeta^2\left(D_\zeta^{\alpha+1}+\sigma^2\right)^{-1}$, therefore, one gets the following modified equation:

$$D_\zeta^2 \upsilon + D_\zeta^2\left(D_\zeta^{\alpha+1}+\sigma^2\right)^{-1}\left(\mu D_\zeta \upsilon + Q\upsilon^3\right)=0. \tag{12}$$

Introducing the new auxiliary parameter $\Omega^2$, so that the Homotopy equation established in the following form:

$$\left(D_\zeta^2+\Omega^2\right)\upsilon=\rho\left\{\Omega^2\upsilon-D_\zeta^2\left(D_\zeta^{\alpha+1}+\sigma^2\right)^{-1}\left(\mu D_\zeta\upsilon+Q\upsilon^3\right)\right\}. \tag{13}$$

where $\rho\in[0,1]$ is an embedding parameter. It is worthwhile to note that as $\rho\to 1$, eq. (11) arises again.

One seeks a solution $\upsilon(\zeta;\rho)$ in the Taylor expansion as follows:

$$\upsilon(\zeta;\rho)=\upsilon_0(\zeta)+\rho\upsilon_1(\zeta)+\rho^2\upsilon_2(\zeta)+\ldots \tag{14}$$

Employing the expansion (14) into eq. (13) and equating like the power of $\rho$ on both sides, the first two unknowns $\upsilon_0(\zeta)$ and $\upsilon_1(\zeta)$ of the expansion (14) is given as

$$\upsilon_0(\zeta)=A\cos(\Omega\zeta+B), \tag{15}$$

where $A$ and $B$ are arbitrary constants.

$$\left(D_\zeta^2+\Omega^2\right)\upsilon_1=\Omega^2\upsilon_0-D_\zeta^2\left(D_\zeta^{\alpha+1}+\sigma^2\right)^{-1}\left(\mu D_\zeta\upsilon_0+Q\upsilon_0^3\right). \tag{16}$$

Inserting the solution (15) into eq. (16), then applying the formulas (8), (9), and (10), one obtains

$$\begin{aligned}\left(D_\zeta^2+\Omega^2\right)\upsilon_1 &= A\Omega^2\left[\frac{\Omega^{2\alpha+2}+\sigma^4+\frac{3}{4}QA^2\sigma^2+\Omega\mu\Omega^{\alpha+1}\cos(\frac{1}{2}\pi\alpha)-\left(2\sigma^2+\frac{3}{4}QA^2\right)\Omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)}{\left(\Omega^{2\alpha+2}+\sigma^4-2\sigma^2\Omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)}\right]\cos(\Omega\zeta+B)\\ &+A\Omega^2\frac{\Omega\mu\left(\Omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)-\sigma^2\right)+\frac{3}{4}QA^2\Omega^{\alpha+1}\cos(\frac{1}{2}\pi\alpha)}{\left(\Omega^{2\alpha+2}+\sigma^4-2\sigma^2\Omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)}\sin(\Omega\zeta+B)+\tfrac{9}{4}A^3\Omega^2Q\frac{\left[\sigma^2\cos(3\Omega\zeta+3B)+(3\Omega)^{\alpha+1}\sin(3\Omega\zeta+3B-\frac{1}{2}\pi\alpha)\right]}{\left((3\Omega)^{2\alpha+2}+\sigma^4-2\sigma^2(3\Omega)^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)}.\end{aligned} \tag{17}$$

To avoid the presence of the secular terms, one eliminates all terms that are represented the coefficients of the trigonometric functions: $\cos(\Omega\zeta+B)$ and $\sin(\Omega\zeta+B)$. It follows the next two conditions:

$$\Omega^{2\alpha+2}+\sigma^4+\frac{3}{4}QA^2\sigma^2+\Omega\mu\Omega^{\alpha+1}\cos(\tfrac{1}{2}\pi\alpha)-\left(2\sigma^2+\frac{3}{4}QA^2\right)\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)=0 \tag{18}$$

$$\Omega\mu\left(\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha)-\sigma^2\right)+\frac{3}{4}QA^2\Omega^{\alpha+1}\cos(\tfrac{1}{2}\pi\alpha)=0. \tag{19}$$

According to the conditions (18) and (19), the finite solution of eq. (17) is given as

$$\upsilon_1(\zeta)=\frac{-9A^3Q\left[\sigma^2\cos(3\Omega\zeta+3B)+(3\Omega)^{\alpha+1}\sin(3\Omega\zeta+3B-\frac{1}{2}\pi\alpha)\right]}{32\left((3\Omega)^{2\alpha+2}+\sigma^4-2\sigma^2(3\Omega)^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)} \tag{20}$$

Subsequently, one can construct higher-order problems easily. The solutions up to the first-order problem are enough to produce a good result. If the series solution as given in eq. (14) is convergent as $\rho=1$, then the periodic approximate solution can be obtained as follows:

$$\upsilon(\zeta)=A\cos(\Omega\zeta+B)-\frac{9A^3Q\left[\sigma^2\cos(3\Omega\zeta+3B)+(3\Omega)^{\alpha+1}\sin(3\Omega\zeta+3B-\frac{1}{2}\pi\alpha)\right]}{32\left((3\Omega)^{2\alpha+2}+\sigma^4-2\sigma^2(3\Omega)^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)} \tag{21}$$

Consequently, the final solution of the original nonlinear damping Duffing oscillator (5) arises when $\alpha=1$, in the form

$$\upsilon(\zeta)=A\cos(\Omega\zeta+B)-\frac{9A^3Q}{32\left(\sigma^2-(3\Omega)^2\right)}\cos(3\Omega\zeta+3B) \tag{22}$$

Therefore, the periodic solution of the NWS eq. (1) with $n=3$ can formulate as

$$v(x,t) = A\cos\big(\Omega(x - t/m) + B\big) + \frac{9\beta A^3}{32m\left(\kappa/m - (3\Omega)^2\right)}\cos\big(3\Omega(x - t/m) + 3B\big) \tag{23}$$

where the frequency $\Omega$ is given in the next Section.

### 3.2 Construction of the Frequency Equation and the Stability Analysis

Combining conditions (18) with (19), across removing the function $\cos(\frac{1}{2}\pi\alpha)$, yields

$$\Omega^2 = -\frac{9Q^2A^4}{16\mu^2} + \frac{3}{4}QA^2\frac{\left(\Omega^{2\alpha+2} + \sigma^4 - 2\sigma^2\Omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)}{\mu^2\left(\Omega^{\alpha+1}\sin(\frac{1}{2}\pi\alpha) - \sigma^2\right)} \tag{24}$$

The limiting case as $\alpha \to 1$, leads to $\sin(\frac{1}{2}\pi\alpha) \to 1$. Consequently, the above frequency equation for $\Omega^2 \neq \sigma^2$ takes the form

$$\Omega^2\left(1 - \frac{3QA^2}{4\mu^2}\right) = -\frac{3QA^2}{4\mu^2}\left(\sigma^2 + \frac{3QA^2}{4}\right) \tag{25}$$

It ought to be noticed that the stability is conducted when the parameter $\Omega^2$ becomes positive, i.e.

$$\frac{3QA^2}{4\mu^2}\left(\frac{3}{4}QA^2 + \sigma^2\right)\left(\frac{3QA^2}{4\mu^2} - 1\right) > 0 \tag{26}$$

The positivity of the above criteria, required

$$Q > \frac{4\mu^2}{3A^2} \tag{27}$$

The above condition (27) is the stability criterion of the classical nonlinear damping Duffing oscillator (5). It's worthy to notice that, the frequency-amplitude eq. (25) and the periodic solution (22) can't be evaluated by using the classical HPM. As appeared from the current examination, it is clear that the role of the fractional calculus to help us to obtain these results. In what follows, the same analytical approach will be applied to analyze the nonlinear NWS model (1) with a negative integer and fractional power.

## 4. A Periodic Solution of a Negative Integer and Fraction Power NWS Wave Model

In the present Section, similar arguments as in the above Section will be applied to obtain an approximate for the nonlinear NWS model (1) with a fraction as well as a negative integer power i.e. at $n = 1/3$ and $n = -3$.
However, when $n$ is a set of these values, one may expose the problem of the appearance of the trigonometric functions $(\cos\zeta)^{1/3}$ and $(\cos\zeta)^{-3}$, which will overcome with by expansion into periodic formulas as shown in the appendix.

### 4.1 A Periodic Analytic Approximate Solution of NWS Wave Model at $n = 1/3$

Now, applying the same procedure as in the above Section in the next NWS wave equation, one gets

$$\frac{\partial v}{\partial t} = m\frac{\partial^2 v}{\partial x^2} + \kappa v - \beta v^{1/3} \tag{28}$$

The next transformed nonlinear fractional derivative damping Duffing oscillator, yields

$$D_\zeta^{\alpha+1}v + \mu D_\zeta v + \sigma^2 v + Qv^{1/3} = 0; \qquad 0 < \alpha \le 1 \tag{29}$$

With the presence of the fractional power, that leads to the existence of this function, which can be expanded into a periodic expansion as presented in the Appendix.
Therefore, the following the final periodic solution for the above fractional derivative eq. (29) can be formulated as:

$$v(\zeta) = A\cos(\Omega\zeta + B) + \frac{16}{81}QA^{\frac{1}{3}} \times \left[\frac{\left[\sigma^2\cos(2\Omega\zeta + 2B) + (2\Omega)^{\alpha+1}\sin(2\Omega\zeta + 2B - \frac{1}{2}\pi\alpha)\right]}{\left((2\Omega)^{2\alpha+2} + \sigma^4 - 2\sigma^2(2\Omega)^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)} - \frac{45}{512}\frac{\left[\sigma^2\cos(3\Omega\zeta + 3B) + (3\Omega)^{\alpha+1}\sin(3\Omega\zeta + 3B - \frac{1}{2}\pi\alpha)\right]}{\left((3\Omega)^{2\alpha+2} + \sigma^4 - 2\sigma^2(3\Omega)^{\alpha+1}\sin(\frac{1}{2}\pi\alpha)\right)}\right]. \tag{30}$$

In accordance of the next two solvability conditions:

$$\Omega\mu\left(\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha) - \sigma^2\right) + \frac{85}{108}QA^{-2/3}\Omega^{\alpha+1}\cos(\tfrac{1}{2}\pi\alpha) = 0 \tag{31}$$

and

$$\Omega^{2\alpha+2} + \sigma^4 + \frac{85}{108}QA^{-2/3}\sigma^2 + \Omega\mu\Omega^{\alpha+1}\cos(\tfrac{1}{2}\pi\alpha) - \left(2\sigma^2 + \frac{85}{108}QA^{-2/3}\right)\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha) = 0 \tag{32}$$

one puts $\alpha = 1$ to formulate the final periodic solution NWS model (28) as:

$$v(x,t) = A\cos\left(\Omega(x - t/m) + B\right) - \frac{16}{81}\frac{\beta A^{1/3}}{m}\left[\frac{\cos\left(2\Omega(x-t/m)+2B\right)}{\left(\kappa/m - (2\Omega)^2\right)} - \frac{45}{512}\frac{\cos\left(3\Omega(x-t/m)+3B\right)}{\left(\kappa/m-(3\Omega)^2\right)}\right] \tag{33}$$

With the next two solvability conditions:

$$\Omega\mu\left(\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha) - \sigma^2\right) + \frac{85}{108}QA^{-2/3}\Omega^{\alpha+1}\cos(\tfrac{1}{2}\pi\alpha) = 0 \tag{34}$$

and

$$\Omega^{2\alpha+2} + \sigma^4 + \frac{85}{108}QA^{-2/3}\sigma^2 + \Omega\mu\Omega^{\alpha+1}\cos(\tfrac{1}{2}\pi\alpha) - \left(2\sigma^2 + \frac{85}{108}QA^{-2/3}\right)\Omega^{\alpha+1}\sin(\tfrac{1}{2}\pi\alpha) = 0 \tag{35}$$

Furthermore, for $\Omega^2 \neq \sigma^2$ the above two solvability conditions can be combined with the next frequency equation:

$$\Omega^2 = \frac{85}{108}\frac{QA^{-2/3}}{\mu^2}\left[-\frac{85}{108}QA^{-2/3} + \left(\Omega^2 - \sigma^2\right)\right] \tag{36}$$

Therefore, by using the original parameters this condition, one can take the form:

$$\frac{85}{108}\beta A^{-2/3}\left(\frac{85}{108}\beta m A^{-2/3} + 1\right)\left(\frac{85}{108}\beta A^{-2/3} + \kappa\right) > 0 \tag{37}$$

**4.2 A Periodic Analytic Approximate Solution of the NWS Wave Model at $n = -3$**

By taking $n = -3$ as a negative integer power into the NWS wave eq. (1), then applying the same approach as before. Consequently, by expanding the function $(\cos\zeta)^{-3}$ that appears in these calculations as presented in the Appendix. The following periodic solution and its stability analysis for the NWS model (1) with $n = -3$ can be formulated in the next forms:

$$v(x,t) = A\cos\left(\Omega(x - t/m) + B\right) + 24\frac{\beta A^{-3}}{m}\left[\frac{\cos\left(2\Omega(x-t/m)+2B\right)}{\left(\kappa/m - (2\Omega)^2\right)} - \frac{315}{384}\frac{\cos\left(3\Omega(x-t/m)+3B\right)}{\left(\kappa/m-(3\Omega)^2\right)}\right] \tag{38}$$

and

$$\frac{117}{4}\beta A^{-4}\left(\frac{117}{4}\beta m A^{-4} - 1\right)\left(\frac{117}{4}\beta A^{-4} + \kappa\right) > 0 \tag{39}$$

## 5. Conclusion

It is well-known that the class of the nonlinear dynamical waves in the mathematical biology, plasma physics, and quantum mechanics has been described by a temporal-spatial NWS model. Therefore, in this proposal, an attempt has been made to achieve an analytical approximate solution of the NWS model with positive and negative integer, and fraction power. A special temporal-spatial transformation was proceeded to acquire a damping Duffing oscillator from the original nonlinear NWS model. Therefore, the transformed nonlinear damping Duffing oscillator is seeking, according to the Riemann-Liouville fractional derivative. In addition to that, a modified HPM is utilized to obtain an analytical periodic approximate solution for this yielding fractional derivative damping Duffing equation. Consequently, for three various values of the power *n*, these steps help in obtaining the periodic solutions and its stability analysis of the original NWS model. Thus, the present method is a promising, powerful and an effective technique to obtain a periodic solution of the time–space nonlinear dynamical wave equation. Furthermore, the success in applying the fractional derivative to the nonlinear damping Duffing oscillator is a very important tool and enable us to analyze different classes of the damping nonlinear equations. Briefly, the present technique will overcome the difficulties to calculate the periodic solution of the damping nonlinear equations without the need of applying the methods used in other publications such as multiple scales or the time-delay.

## Acknowledgments

The author is thankful to all reviewers for their valuable, encouraging comments as well as constructive suggestions to improve the original article.

## Conflict of Interest

The author declared that no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

## Funding

The author received no financial support of the research, authorship, and/or publication of this article.

## References

[1] Polyanin, A.D., Zaitsev, V.F., *Handbook of Nonlinear Partial Differential Equations*, Chapman & Hall/CRC, Boca Raton, 2004.
[2] Hariharan, G., An efficient Legendre wavelet-based approximation method for a few Newell–Whitehead and Allen–Cahn equations, *Journal of*

*Membrane Biology*, 247 (5), 2014, 371–380.
[3] Newell, A.C., Whitehead, J.A., Finite bandwidth, finite amplitude convection, *Journal of Fluid Mechanics*, 38(2), 1969, 279-303.
[4] Segel, L.A., Distant side-walls cause slow amplitude modulation of cellular convection, *Journal of Fluid Mechanics*, 38(1), 1969, 203-224.
[5] Zellal, M., Belghaba, K., Applications of homotopy perturbation transform method for solving Newell-Whitehead-Segel equation, *General Letters in Mathematics*, 3(1), 2017, 35-46.
[6] Nourazar, S.S. , Soori, M., Nazari-Golshan, A., On the exact solution of Newell-Whitehead-Segel equation using the homotopy perturbation method, *Australian Journal of Basic and Applied Sciences*, 5(8), 2011, 1400-1411.
[7] Jassim, H.K., Homotopy perturbation algorithm using Laplace transform for Newell-Whitehead-Segel equation, *International Journal of Advances in Applied Mathematics and Mechanics*, 2(4), 2015, 8-12.
[8] Valls, C., Algebraic traveling waves for the generalized Newell–Whitehead–Segel equation, *Nonlinear Analysis: Real World Applications*, 36, 2017, 249-266.
[9] Korkmaz, A., Complex wave solutions to mathematical biology models I: Newell–Whitehead–Segel and Zeldovich equations, *Journal of Computational and Nonlinear Dynamics*, 13, 2018, 081004.
[10] Haller, E., Hart, R., Mark, M.J., Danzl, J.G., ReichsÖllner, L., Gustavsson, M., Dalmonte, M., Pupillo, G., Nägerl, H.C., Pinning quantum phase transition for a Luttinger liquid of strongly interacting bosons, *Nature*, 466, 2010, 597-601.
[11] Schweigler, T., Kasper, V., Erne, S., Mazets, I., Rauer, B., Cataldini, F., Langen, T., Gasenzer, T., Berges, J., Schmiedmayer, J., Experimental characterization of a quantum many-body system via higher-order correlations, *Nature*, 545, 2017, 323-335.
[12] Prakash, A., Kumar, M., He's variational iteration method for the solution of nonlinear Newell-Whitehead-Segel equation, *Journal of Applied Analysis and Computation*, 6(3), 2016, 738–748.
[13] Akinlabi, G.O., Edeki, S.O., Perturbation iteration transform method for the solution of Newell-Whitehead-Segel model equations, *Journal of Mathematics and Statistics*, 13(1), 2017, 24-29.
[14] Patade, J., Bhalekar, S., Approximate analytical solutions of Newell-Whitehead-Segel equation using a new iterative method, *World Journal of Modelling and Simulation*, 11(2), 2015, 94-103.
[15] Aswhad, A.A., Jaddoa, A.F., The approximate solution of Newell-Whitehead-Segel and Fisher equations using the Adomian decomposition method, *Al-Mustansiriyah Journal of Science*, 25(4), 2014, 45-56.
[16] Prakash, A., Verma, V., Numerical method for fractional model of Newell-Whitehead-Segel equation, *Frontiers in Physics*, 7, 2019, Article 15.
[17] Mahgoub, M.M.A., Sedeeg, A.K.H., On The Solution of Newell-Whitehead-Segel equation, *American Journal of Mathematical and Computer Modelling*, 1(1), 2016, 21-24.
[18] Pue-on, P., Laplace Adomian decomposition method for solving Newell-Whitehead-Segel equation, *Applied Mathematical Sciences*, 7(132), 2013, 6593 – 6600.
[19] Saravanan, A., Magesh, N., A comparison between the reduced differential transform method and the Adomian decomposition method for the Newell–Whitehead–Segel equation, *Journal of the Egyptian Mathematical Society*, 21, 2013, 259-265.
[20] Nadeem, M., Yao, S.W., Parveen, N., Solution of Newell-Whitehead-Segel equation by variational iteration method with He's polynomials, *Journal of Mathematics and Computer Science*, 20, 2020, 21–29.
[21] Hilal, N., Injrou, S., Karroum, R., Exponential finite difference methods for solving Newell–Whitehead–Segel equation, *Arabian Journal of Mathematics*, 9, 2020, 367–379.
[22] Almousa, M., Al-Zuhairi, I., Al-Qudah, Y., Qoqazeh, H., Mahgoub Adomian decomposition method for solving Newell-Whitehead-Segel equation, *International Journal of Mathematics and Statistics Invention*, 8(1), 2020, 22-24.
[23] Ahmad, H., Seadawy, A.R., Khan, T.A., Thounthong, P., Analytic approximate solutions for some nonlinear parabolic dynamical wave equations, *Journal of Taibah University for Science*, 14(1), 2020, 346–358.
[24] Saha, A., Das, A., Dynamical behavior of nonlinear wave solutions of the generalized Newell-Whitehead-Segel equation, *International Journal of Modern Physics C*, 31(4), 2020, 2050059.
[25] Edeki, S.O., Ogundile, O.P., Osoba, B., Adeyemi, G.A., Egara, F.O., Ejoh, S.A., Coupled FCT-HP for analytical solutions of the generalized time fractional Newell-Whitehead-Segel equation, *WSEAS Transactions on Systems and Control*, 13, 2018, 266-274.
[26] Saadeh, R., Alaroud, M., Al-Smadi, M., Ahmad, R.R., Din, U.K.S., Application of fractional residual power series algorithm to solve Newell–Whitehead–Segel equation of fractional order, *Symmetry*, 11, 2019, 1431.
[27] Saberi, E., Hejazi, S.R., Motamednezhad, A., Lie symmetry analysis, conservation laws and similarity reductions of Newell-Whitehead-Segel equation of fractional order, *Journal of Geometry and Physics*, 135, 2019, 116-128.
[28] Ali, M., Alquran, M. Jaradat, I., Asymptotic-sequentially solution style for the generalized Caputo time-fractional Newell–Whitehead–Segel system, *Advances in Difference Equations*, 2019, 2019, 70.
[29] Kumar, D., Sharma, R.P., Numerical approximation of Newell-Whitehead-Segel equation of fractional order, *Nonlinear Engineering*, 5(2), 2016, 81–86.
[30] A. Prakash, M. Goyal and S. Gupta, Fractional variational iteration method for solving time-fractional Newell-Whitehead-Segel equation, *Nonlinear Engineering*, 8, 2019, 164–171.
[31] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.
[32] Samko, S.G., Kilbas, A.A., Marichev, D.I., *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Switzerland; Philadelphia, Pa., USA, 1993.
[33] Guy, J., Modeling fractional stochastic systems as nonrandom fractional dynamics driven by Brownian motions, *Applied Mathematical Modelling*, 32, 2008, 836–859.
[34] Qian, D.L., Li, C.P., Agarwal, R.P., Wong, P.J.Y., Stability analysis of fractional differential system with Riemann-Liouville derivative, *Mathematical and Computer Modelling*, 52, 2010, 862–874.
[35] Zhao, Z.G., Guo, Q., Li, C.P., A fractional model for the allometric scaling laws, *The Open Applied Mathematics Journal*, 2, 2008, 26–30.
[36] Li, C., Qian, D., Chen, Y.Q., On Riemann-Liouville and Caputo Derivatives, *Discrete Dynamics in Nature and Society*, 2011, 2011, Article ID 562494.
[37] Valério, D., Trujillo, J.J., Rivero, M., Machado, J.A.T., Baleanu, D., Fractional calculus: A survey of useful formulas, *The European Physical Journal Special Topics*, 222, 2013, 1827-1846.
[38] Shadab, M., Khan, M.F., Lopez-Bonilla, J.L., A new Riemann–Liouville type fractional derivative operator and its application in generating functions, *Advances Difference Equations*, 2018, 2018, 167.
[39] El-Dib, Y.O., Stability approach of a fractional-delayed Duffing oscillator, *Discontinuity, Nonlinearity, and Complexity*, 9, 2020, 367-376.
[40] Llibre, J., Martínez, Y.P., Valls, C., On the global dynamics of a three-dimensional forced-damped differential system, *Journal of Nonlinear Mathematical Physics*, 27(3), 2020, 414-428.
[41] Padovan, J., Sawicki, J.T., Nonlinear vibration of fractionally damped systems, *Nonlinear Dynamics*, 16, 1998, 321–336.
[42] Liao, H., Optimization analysis of Duffing oscillator with fractional derivatives, *Nonlinear Dynamics*, 79, 2015, 1311–1328.
[43] El-Dib, Y.O., Modified multiple scale technique for the stability of the fractional delayed nonlinear oscillator, *Pramana- Journal of Physics*, 94, 2020, 56.
[44] Shen, Y., Yang, S., Xing, H., Gao, G., Primary resonance of Duffing oscillator with fractional-order derivative, *Communications in Nonlinear Science and Numerical Simulation*, 17(7), 2012, 3092–3100.
[45] Jiménez, S., González, J.A., Vázquez, L., Fractional Duffing's equation and geometrical resonance, *International Journal of Bifurcation and Chaos*, 23, 2013, 1350089-1—1350089-13.
[46] Ejikeme, C.L., Oyesanya, M.O., Agbebaku, D.F., Okofu, M.B., Solution to nonlinear Duffing oscillator with fractional derivatives using homotopy analysis method (HAM), *Global Journal of Pure and Applied Mathematics*, 14(10), 2018, 1363–1388.
[47] El-Dib, Y.O., Elgazery, N.S., Effect of fractional derivative properties on the periodic solution of the nonlinear oscillations, *Fractals*, 2020, https://doi.org/10.1142/S0218348X20500954.
[48] Shen, Y., El-Dib, Y.O., A periodic solution of the fractional sine-Gordon equation arising in architectural engineering, *Journal of Low Frequency Noise, Vibration and Active Control*, 2020, https://doi.org/10.1177/ 1461348420917565.
[49] Zhurov, A.I., Polyanin, A.D., Symmetry reductions and new functional separable solutions of nonlinear Klein–Gordon and telegraph type equations, *Journal of Nonlinear Mathematical Physics*, 27(2), 2020, 227-242.
[50] El-Dib, Y.O., Multiple scales homotopy perturbation method for nonlinear oscillators, *Nonlinear Science Letters A: Mathematics, Physics and Mechanics*,

8 (4), 2017, 352–364.
[51] El-Dib, Y.O., Stability analysis of a strongly displacement time-delayed Duffing oscillator using multiple scales homotopy perturbation method, *Journal of Applied and Computational Mechanics*, 4(4), 2018, 260-274.
[52] Oldham, K.B., Spanier, J., *The Fractional Calculus*, Academic Press, New York, 1999.
[53] Podlubny, I., *Fractional Differential Equation*, Academic Press, San Diego, 1999.
[54] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Application of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[55] De la Sen, M., Basic Riemann-Liouville and Caputo impulsive fractional calculus, *Applied Mathematical Sciences*, 5(15) , 2011, 717 – 727.
[56] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer, New York, 2010.
[57] Ortigueira, M.D., Machado, J.T., Trujillo, J.J., Fractional derivatives and periodic functions, *International Journal of Dynamics and Control*, 5(1), 2015, 1-7.
[58] Baishya, C., Solution of Newell-Whitehead-Segel equation with power law nonlinearity and time dependent coefficients, *International Journal of Advances in Applied Mathematics and Mechanics*, 3(2), 2015, 59 – 64.
[59] Èidel'man, S.D., On fundamental solutions of parabolic systems, *Matematicheskii Sbornik*, 38(80), 1956, 51–92.
[60] B. Zhu and B. Han, Existence and Uniqueness of Mild Solutions for Fractional Partial Integro-Differential Equations, *Mediterranean Journal of Mathematics*, 17, 2020, 113.
[61] Ouyang, Z., Existence and uniqueness of the solutions for a class of nonlinear fractional order partial differential equations with delay, *Computers and Mathematics with Applications*, 61, 2011, 860–870.
[62] El-Borai, M.M., El-Nadi, K.El-S., Ahmed, H.M., El-Owaidy, H.M., Ghanem, A.S., Sakthivel, R., Existence and stability for fractional parabolic integro-partial differential equations with fractional Brownian motion and nonlocal condition, *Cogent Mathematics and Statistics*, 5, 2018, 1460030.
[63] He, J.H., Homotopy perturbation technique, *Computer Methods in Applied Mechanics and Engineering*, 178, 1999, 257–262.
[64] He, J.H., The simplest approach to nonlinear oscillators, *Results in Physics*, 15, 2019, 102546.
[65] He, J.H., The simpler, the better: analytical methods for nonlinear oscillators and fractional oscillators, *Journal of Low Frequency Noise, Vibration and Active Control*, 38, 2019, 1252–1260.
[66] He, J.H., Jin, X., A short review on analytical methods for the capillary oscillator in a nanoscale deformable tube, *Mathematical Methods in the Applied Sciences*, 2020, 1-8, https://doi.org/10.1002/mma.6321.

## Appendix

Herein, the HPM; for instance, see Refs. [53-66] is applied to obtain an approximate expansion of the function $(\cos\zeta)^n$ in the periodic forms: For the two fractional cases ($n = 1/3$) and negative integer ($n = -3$) power as follows:

- The function $(\cos\zeta)^{1/3}$ that appears in Section 4.1 may be rewritten via the HPM as follows:

$$(\cos\zeta)^{1/3} = \lim_{\rho\to 1}\left[1+\rho(\cos\zeta-1)\right]^{1/3}; \quad 0 \le \rho \le 1$$

Expand for the parameter $\rho$ as follows:

$$(\cos\zeta)^{\frac{1}{3}} = \lim_{\rho\to 1}\left[\left(1-\frac{1}{3}\rho-\frac{1}{9}\rho^2-\frac{5}{81}\rho^3-\ldots\right)+\left(\frac{1}{3}\rho+\frac{2}{9}\rho^2+\frac{5}{27}\rho^3+\ldots\right)\cos\zeta-\left(\frac{1}{9}\rho^2+\frac{5}{27}\rho^3+\frac{40}{243}\rho^4+\ldots\right)\cos^2\zeta+\left(\frac{5}{81}\rho^3+\frac{40}{243}\rho^4+\frac{220}{729}\rho^5+\ldots\right)\cos^3\zeta+\ldots\right]$$

If one approximates the above series to the cubic order for the parameter $\rho$ and take the limit as $\rho \to 1$. Therefore,

$$(\cos\zeta)^{1/3} \cong \frac{40}{81}+\frac{20}{27}\cos\zeta-\frac{8}{27}\cos^2\zeta+\frac{5}{81}\cos^3\zeta-\ldots$$

- Furthermore, the HPM is utilized to refine the function $(\cos x)^{-3}$ that appears in Section 4.2 in a periodic formula as:

$$(\cos\zeta)^{-3} = \lim_{\rho\to 1}\left[1+\rho(\cos\zeta-1)\right]^{-3}; \quad 0 \le \rho \le 1$$

Following the above procedure, one finds:

$$(\cos\zeta)^{-3} \cong 38-\tfrac{117}{4}\cos\zeta+18\cos 2\zeta-\frac{35}{2}\cos 3\zeta+\ldots$$

## ORCID iD

Nasser S. Elgazery https://orcid.org/0000-0003-4691-2526



How to cite this article: Elgazery N.S., A Periodic Solution of the Newell-Whitehead-Segel (NWS) Wave Equation via Fractional Calculus, *J. Appl. Comput. Mech.*, 6(SI), 2020, 1293–1300. https://doi.org/10.22055/JACM.2020.33778.2285