J. Appl. Comput. Mech., 6(SI) (2020) 1260-1269
DOI: 10.22055/JACM.2020.33461.2229

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

Application of Atangana-Baleanu Fractional Derivative to Carbon Nanotubes Based Non-Newtonian Nanofluid: Applications in Nanotechnology

Kashif Ali Abro[1], Muzaffar Hussain Laghari[2], J.F. Gómez-Aguilar[3,4]

[1] Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan

[2] Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan

[3] Departamento de Ingeniería Electrónica, CONACyT-Tecnológico Nacional de México/CENIDET, Interior Internado Palmira S/N Col. Palmira, C.P. 62490, Cuernavaca Morelos, México, Email: jose.ga@cenidet.tecnm.mx

[4] Universidad Virtual CNCI. Av. Ruiz Cortines No. 5901, Col. Valle Verde, 1er. Sector. Monterrey, Nuevo León, C.P. 64360, México

Received May 02 2020; Revised June 09 2020; Accepted for publication June 09 2020.
Corresponding author: J.F. Gómez-Aguilar (jose.ga@cenidet.tecnm.mx)

Abstract: Single and multi-walled carbon nanotubes (SWCNTs & MWCNTs) comprise a large group of nanometer-thin hollow fibrous nanomaterials having physico-chemical characteristics like atomic configuration, length to diameter ratios, defects, impurities and functionalization. This manuscript is devoted for the analysis of carbon nanotubes based non-Newtonian nanofluid suspended in ethylene glycol taken as base fluid. The problem is modeled through modern method of fractional calculus namely Atangana-Baleanu fractional derivative and then solved analytically by invoking Laplace transform. The analytic solutions are established for the temperature and velocity distribution and expressed in terms of special function. The graphical results are depicted through computational software Mathcad and discussed for carbon nanotubes with various embedded parameters. An interesting comparison is explored graphically between single and multi-walled carbon nanotubes subject to the single and multi-walled carbon nanotubes are suspended in ethylene glycol. The several similarities and differences suggested that carbon nanotubes are accelerated and decelerated, while for unit time $t = 1$s, carbon nanotubes have identical velocities with and without fractional approach.

Keywords: Carbon Nanotubes; Atangana-Baleanu Fractional derivative; Analytical approach; Fox-H function.

1. Introduction

Nowadays the discoveries of different forms of carbon nanostructures have motivated research on their applications in various fields, hence these discoveries of carbon nanotubes (CNT) in 1991 opened up a new era in materials science. The fascinating properties and incredible structures of carbon nanotubes (CNT) are enumerated as (i) carbon nanotubes are good electron field emitters, (ii) carbon nanotubes have high electrical and thermal conductivity, (iii) carbon nanotubes have very high tensile strength, (iv) carbon nanotubes have a low thermal expansion coefficient, (v) carbon nanotubes are very elastic approximately 18% elongation to failure, (vi) carbon nanotubes are highly flexible which can be bent considerably without damage, (vii) carbon nanotubes are at least 100 times stronger than steel and only one-sixth as heavy, (viii) carbon nanotubes are unique because the bonding among the atoms is very strong and the tubes can have extreme aspect ratios, (ix) carbon nanotubes have many structures, differing in thickness, number of layers and length, (x) carbon nanotubes are thin and cylindrical shaped with diameter ranges 0.7-50 nm and many other electronic, magnetic and mechanical properties as well. These splendid properties of carbon nanotubes have great significance and applications. For instance, carbon nanotubes' thermal conductivity, carbon nanotubes' field emission, carbon nanotubes' conductive adhesive, carbon nanotubes' energy storage, carbon nanotubes' conductive properties, carbon nanotubes' fibers and fabrics, carbon nanotubes' biomedical applications, carbon nanotubes' thermal materials, carbon nanotubes' air and water Filtration, carbon nanotubes' catalyst supports, Molecular electronics based on carbon nanotubes, carbon nanotubes' structural applications and several others. Carbon nanotubes are categorized as single, double and multi walled as shown in Fig. 1.

Choi [1] gave a concept about nanomaterial in fluid mechanics to enhance the convective heat transfer coefficient and thermal conductivity of base liquid. Ramasubramaniam et al. [2] investigated the electrical conductivities of two different types of polymer composite namely PPE-SWNTs/polycarbonate and PPE-functionalized SWNTs/polystyrene composites. Xue [3] presented the validity of the transport properties of the carbon nanotubes-based composites. Berber et al. [4] determined the thermal conductivity lambda of carbon nanotubes having dependence on temperature.

Published online: August 25 2020

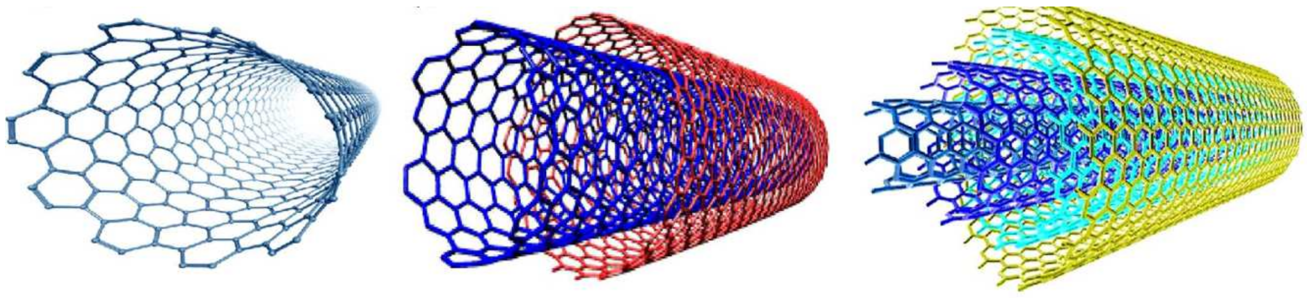

Fig. 1. Structure of carbon nanotubes.

Hone et al. [5] synthesized nanotube-based composite materials and measured their thermal conductivity. Jiang et al. [6] presented the study for building a model for predicting the thermal conductivities of carbon nanotubes nanorefrigerants and to test thermal conductivity characteristics of carbon nanotubes nanorefrigerants. Tan and Mieno [7] calculated the temperature distribution of the buffer gas and the cooling rate of carbon clusters through numerical analysis on heat convection in the synthesis of single-walled carbon nanotubes. Mayer et al. [8] studied a hot problem of carbon nanotubes and heat transfer in presence of aqueous suspensions experimentally in which they suggested the increase in viscosity was four times the increase in the thermal conductivity. Haq et al. [9] investigated the water functionalized carbon nanotubes squeezing flow between two parallel discs in which comparison among carbon nanotubes are analyzed for local Nusselt number and skin friction coefficient. Haq et al. [10] observed the effects of volume fraction of carbon nanotubes (CNTs) and magnetohydrodynamics (MHD) on the flow and heat transfer over a stretching sheet. They explored the comparisons for three models namely (i) Hamilton and Crosser model, (ii) Maxwell model and (iii) Xue model for the thermal conductivity. Ellahi et al. [11] solved the nonlinear partial differential equations for natural convection boundary layer magnetohydrodynamic flow with variable wall temperature based on nanolayer single and multi-wall carbon nanotubes in salt-water. In brevity, the studies on carbon nanotubes [12-16], heat and mass transfer [17-24], nanofluids [25-31], nanoparticles [32-33], modern fractional methods [34-44] and nanotubes in Microfluidic devices [52-61] can be continued but we end here with latest attempts therein. Inspiring by above discussed literature, our purpose is to devote an analysis of carbon nanotubes based Casson nanofluid suspended in ethylene glycol. The governing equations are transferred through modern technique of fractional derivatives namely Atangana-Baleanu fractional operator and then investigated analytically using integral transform. The analytic calculations are investigated for the expression of temperature and velocity distribution and then written in the layout of Fox-**H** special function. The graphical results are depicted for the discussion of carbon nanotubes with various embedded parameters. A comparison is explored graphically between single and multi-walled carbon nanotubes when they are suspended in ethylene glycol.

2. Modeling of Problem via Atangana-Baleanue Fractional Derivative

The Cauchy stress tensor of Casson fluid in terms of constitutive equations is described as [45-46]

$$\tau_{ij}=\begin{cases}2\left(\dfrac{p_y}{\sqrt{2\pi}}+\mu\right)e_{ij}, & \pi>\pi_c\\[2ex] 2\left(\dfrac{p_y}{\sqrt{2\pi_c}}+\mu\right)e_{ij}, & \pi>\pi_c\end{cases}. \tag{1}$$

Assume an unsteady flow of Casson nanofluid consisting of carbon nanotubes occupying a semi-finite space $y>0$ with constant kinematic viscosity. An electrically conducting plate is assumed with a transverse magnetic field of strength B_0 that is applied to the plate in a perpendicular direction. In order to have negligence of applied magnetic field, a small amount of magnetic Reynolds number is considered. The Casson nanofluid consisting of carbon nanotubes is saturated with porous plate. The carbon nanotubes and base fluid are under the assumptions of no slip occurrence and in thermal equilibrium. The fluid and a plate are at rest with constant temperature T_∞ subject to the condition when $\tau=0$. A plate has nonlinear velocity $w=\Omega_0 H(t)\tau^a$ over the bounding wall when $\tau=0^+$ and temperature of plate is raised up to T_w which is thereafter preserved as a constant [17, 47-48]

$$\frac{\partial w(\xi,\tau)}{\partial\tau}=\frac{1}{\rho_{nf}}\left\{\frac{\mu_{nf}\partial^2 w(\xi,\tau)}{\partial\xi^2}+\frac{\mu_{nf}}{\gamma}\frac{\partial^2 w(\xi,\tau)}{\partial\xi^2}+\frac{\mu_{nf}w(\xi,\tau)}{k}-\sigma_{nf}B_0^2 w(\xi,\tau)+g(\rho\beta)_{nf}\left(T(\xi,\tau)-T_\infty(\xi,\tau)\right)\right\}, \tag{2}$$

$$\frac{\partial T(\xi,\tau)}{\partial\tau}=\frac{k_{nf}}{(\rho c_p)_{nf}}\frac{\partial^2 T(\xi,\tau)}{\partial\xi^2}. \tag{3}$$

The appropriate initial and boundary conditions for equations (1-2) are

$$w(\xi,0)=0,\quad T(\xi,\tau)=T_\infty;\quad \tau=0,\quad \text{for all}\ \ \xi>0, \tag{4}$$

$$w(0,\tau)=u_0 H(\tau)\tau^a,\quad T(0,\tau)=1;\quad \tau\ge 0, \tag{5}$$

$$w(\infty,\tau)=0,\quad T(\infty,\tau)=0;\qquad \tau>0, \tag{6}$$

Table. 1. Thermo-physical properties of Ethylne Glycol and nanoparticles (CNTs)

Base Fluid/Nanoparticles	ρ (Kg/m^3)	C_p (J/Kg K)	k (W/m.K)
Ethylene Glycol	1.115	0.58	0.1490
SWCNTs	2600	425	6600
SWCNTs	1600	796	3000

where $w(\xi,\tau), T(\xi,\tau)$, $\gamma=\mu\sqrt{2\pi_c}\,/\,p_y$ are the velocity field, the temperature distribution, Casson-nanofluid parameter of Casson-nanofluid respectively. While ρ_{nf}, μ_{nf}, β_{nf}, k_{nf} and $(c_p)_{nf}$ are density, dynamic viscosity, thermal expansion coefficient, thermal conductivity and heat capacitance of nanofluid. In this continuation, Maxwell theory was proposed by Xue [3] in which rotational elliptical nanotubes were considered with compensating the effects of the space distribution on carbon nanotubes via very large axial ratio can be described as

$$\rho_{nf}=\phi\rho_{\mathrm{CNTs}}+(1-\phi)_{\rho f},\mu_{nf}=\frac{\mu_f}{(1-\phi)^{25}},\beta_{nf}=\frac{\phi(\rho\beta)_{\mathrm{CNTs}}+(\rho\beta)_f(1-\phi)}{\rho_{nf}},$$

$$(\rho C_p)_{nf}=\phi(\rho C_p)_{\mathrm{CNTs}}+(\rho C_p)_f(1-\phi),\tag{7}$$

$$\frac{k_{nf}}{k_f}=\frac{1-\phi+\dfrac{2k_{\mathrm{CNTs}}\phi}{k_{\mathrm{CNTs}}-k_f}\ln\dfrac{k_f+k_{\mathrm{CNTs}}}{2k_f}}{1-\phi+\dfrac{2k_f\phi}{k_{\mathrm{CNTs}}-k_f}\ln\dfrac{k_f+k_{\mathrm{CNTs}}}{2k_f}},\frac{\sigma_{nf}}{\sigma_f}=\left\{1+\frac{3\left(\dfrac{\sigma_{\mathrm{CNTs}}}{\sigma_f}-1\right)\phi}{\left(\dfrac{\sigma_{\mathrm{CNTs}}}{\sigma_f}+2\right)-\left(\dfrac{\sigma_{\mathrm{CNTs}}}{\sigma_f}-1\right)\phi}\right\}$$

In order to determine the dimensionless heat transfer rates and the thermal conductivity of nanofluid, we need to give subscript as CNTs are referred as carbon nanotubes, f as fluid and ϕ solid volume fraction of Casson nanofluid.

Employing the suitable dimensionless quantities in equations (2-3) as described below

$$w^*=\frac{w}{u_0},\tau^*=\frac{\tau}{\tau_0},\xi^*=\frac{\xi}{\sqrt{\tau_0 \upsilon}},\ \theta=\frac{T-T_\infty}{T_w-T_\infty}.\tag{8}$$

The recovered governing partial differential equations are

$$\frac{\partial\theta(\xi,\tau)}{\partial\tau}=\frac{\lambda}{\Pr\Re_4}\frac{\partial^2\theta(\xi,\tau)}{\partial\xi^2},\tag{9}$$

$$\frac{\partial w(\xi,\tau)}{\partial\tau}=\frac{1}{\Re_1}\left(\frac{\partial^2 w(\xi,\tau)}{\partial\xi^2}+\frac{1}{\gamma}\frac{\partial^2 w(\xi,\tau)}{\partial\xi^2}\right)-\frac{\mathrm{M}}{\Re_2}w(\xi,\tau)-\frac{\phi_1}{K}w(\xi,\tau)+Gr\Re_3\theta(\xi,\tau)\tag{10}$$

subject to the initial and boundary conditions defined as

$$w(\xi,0)=0,\quad T(\xi,0)=0;\quad \tau=0,\quad \text{for all}\ \ \xi>0,\tag{11}$$

$$w(0,\tau)=H(\tau)\tau^a,\quad T(0,\tau)=1;\quad \tau\geq 0,\tag{12}$$

$$w(\infty,\tau)=0,\quad T(\infty,\tau)=0;\quad \tau>0.\tag{13}$$

Meanwhile,

$$\mathrm{K}=\frac{u_0{}^2k_1}{\upsilon_f^2},\ \mathrm{M}=\frac{\mathrm{B}_0^2\sigma_f\upsilon_f}{u_0{}^2\rho_f},\ \Pr=\frac{(\rho C_p)_f\,\upsilon_f}{k_f},\ Gr=\frac{g(T_w-T_\infty)\beta_f\upsilon_f}{u_0{}^3},\tag{14}$$

are the permeability parameter, magnetic parameter, Prandtl number and Grashof number respectively, and the following quantities are sated as

$$\Re_1=(1-\phi)^{2.5}\left\{(1-\phi)+\phi\frac{\rho_{\mathrm{CNTs}}}{\rho_f}\right\},\ \Re_2=1+\frac{3\left(\dfrac{\sigma_{\mathrm{CNTs}}}{\sigma_f}-1\right)\phi}{\left(\dfrac{\sigma_{\mathrm{CNTs}}}{\sigma_f}+2\right)-\phi\left(\dfrac{\sigma_{\mathrm{CNTs}}}{\sigma_f}-1\right)}(1-\phi)+\phi\frac{\rho_{\mathrm{CNTs}}}{\rho_f},$$

$$\Re_3=\left\{\frac{(1-\phi)+\phi\dfrac{(\rho\beta)_{\mathrm{CNTs}}}{(\rho\beta)_f}}{(1-\phi)+\phi\dfrac{\rho_{\mathrm{CNTs}}}{\rho_f}}\right\},\ \Re_4=\left\{(1-\phi)+\phi\frac{(\rho C_p)_{\mathrm{CNTs}}}{(\rho C_p)_f}\right\},\lambda=\frac{k_{nf}}{k_f}.\tag{15}$$

3. Solution of the Problem

3.1 Temperature distribution through Atangana-Baleanu time fractional operator

In order to model the governing equation (9) of the temperature distribution via Atangana-Baleanu time fractional operator, we get fractional partial differential equation in new transferred operator defined as

$$^{AB}\left(\frac{\partial^{\alpha}\theta(\xi,\tau)}{\partial\tau^{\alpha}}\right)=\frac{\lambda}{\Pr\Re_4}\frac{\partial^2\theta(\xi,\tau)}{\partial\xi^2}. \tag{16}$$

Here, $^{AB}(\partial^{\alpha}\theta(\xi,\tau)/\partial\tau^{\alpha})$ is the Atangana-Baleanu fractional differential operator of order $\alpha=[0,1]$ defined as [37-38]

$$^{AB}\left(\frac{\partial^{\alpha}\theta(\xi,\tau)}{\partial\tau^{\alpha}}\right)=M(\alpha)\int_0^{\tau}\mathbf{E}_{\alpha}\left(\frac{-\alpha(z-\tau)^{\alpha}}{1-\alpha}\right)\frac{\theta'(\xi,\tau)}{1-\alpha}d\tau. \tag{17}$$

Here, $M(\alpha)=M(0)=M(1)=1$ is so called normalization function. Using Laplace transform on equation (16) and considering initial and boundary conditions (11-13), we have an expression in terms of partial differential equation as

$$\left(\frac{\partial}{\partial\xi^2}-\frac{q^{\alpha}\beta\Pr\Re_4}{\lambda(q^{\alpha}+\alpha\beta)}\right)\bar{\theta}(\xi,q)=0. \tag{18}$$

For the sake of simplicity manuscript, we define the letting parameter as $\beta=1/1-\alpha$. The solution of differential equation (18) is simplified using equations (11-13), we obtain

$$\bar{\theta}(\xi,q)=\frac{1}{q}e^{-\xi\sqrt{\frac{q^{\alpha}\beta\Pr\Re_4}{\lambda(q^{\alpha}+\alpha\beta)}}}. \tag{19}$$

The equivalent expression of equation (19) was investigated by employing suitable generalized series for fulfilling the imposed conditions on governing equations, we recovered

$$\bar{\theta}(\xi,q)=\frac{1}{q}+\sum_{\eta_1=1}^{\infty}\frac{1}{\eta_1!}\left(-\xi\sqrt{\frac{\beta\Pr\Re_4}{\lambda}}\right)^{\eta_1}\sum_{\eta_2=0}^{\infty}\frac{(-\alpha\beta)^{\eta_2}\Gamma\left(\eta_2+\frac{\eta_1}{2}\right)}{\eta_2!\Gamma\left(\frac{\eta_1}{2}\right)q^{\alpha\eta_2+2}}. \tag{20}$$

where η_1 and η_2 are the summation parameters. Expressing equation (20) by means of special function and inverting it via Laplace transform, we have solution for temperature distribution in terms of Mittag-Leffler function [50]

$$\theta(\xi,\tau)=1+\sum_{\eta_1=1}^{\infty}\frac{1}{\eta_1!}\left(-\xi\sqrt{\frac{\beta\Pr\Re_4}{\lambda}}\right)^{\eta_1}\mathbf{M}_{\alpha,1}^{\eta_1/2}\left(-\alpha\beta\tau^{\alpha}\right). \tag{21}$$

Equation (21) justifies the imposed initial and boundary conditions as described in equations (11-13).

3.2 Velocity distribution through Atangana-Baleanu time fractional operator

In order to model the governing equation (10) of the velocity distribution via Atangana-Baleanu time fractional operator, we get fractional partial differential equation in new transferred operator defined as

$$^{AB}\left(\frac{\partial^{\alpha}w(\xi,\tau)}{\partial\tau^{\alpha}}\right)=\frac{1}{\Re_1}\left(\frac{\partial^2 w(\xi,\tau)}{\partial\xi^2}+\frac{1}{\gamma}\frac{\partial^2 w(\xi,\tau)}{\partial\xi^2}\right)-\frac{\mathrm{M}}{\Re_2}w(\xi,\tau)-\frac{\phi_1}{K}w(\xi,\tau)+Gr\Re_3\theta(\xi,\tau) \tag{22}$$

Using Laplace transform on equation (22) and considering initial and boundary conditions (11-13), we have an expression in terms of partial differential equation as

$$\left\{\frac{\partial}{\partial\xi^2}-\left(\frac{1}{\Re_1+\Re_1\gamma}\right)\left(\frac{q^{\alpha}\beta}{(q^{\alpha}+\alpha\beta)}-\frac{\mathrm{M}}{\Re_2}-\frac{\Re_1}{k}\right)\right\}\bar{w}(\xi,q)=-\frac{Gr\Re_3\bar{\theta}(\xi,q)}{\left(\frac{1}{\Re_1+\Re_1\gamma}\right)}. \tag{23}$$

The solution of differential equation (23) is simplified using equations (11-13), we obtain

$$\bar{w}(\xi,q)=\frac{p!}{q^{p+1}}e^{-\xi\sqrt{\left(\frac{1}{\Re_1+\Re_1\gamma}\right)\left(\frac{q^{\alpha}\beta}{(q^{\alpha}+\alpha\beta)}-\frac{\mathrm{M}}{\Re_2}-\frac{\Re_1}{k}\right)}}-\frac{Gr\Re_5(q^{\alpha}+\alpha\beta)}{(q^{\alpha}+\Re_8)}e^{-\xi\sqrt{\frac{q^{\alpha}\beta\Pr\Re_4}{\lambda(q^{\alpha}+\alpha\beta)}}}, \tag{24}$$

Here,

$$\Re_5=\lambda\Re_1\Re_2\Re_3\left(1+\frac{1}{\gamma}\right),\quad \Re_6=\left(\frac{\Pr k\Re_1\Re_2\Re_3}{\gamma}-\beta\lambda k\Re_1\Re_2-M\lambda-\Re_1\Re_2\lambda+\Re_1\Re_2\Re_3 k\Pr\right)$$

$$\Re_7=-\left(M\lambda\alpha\beta+\lambda\alpha\beta\Re_1\Re_2\right),\quad \Re_8=\frac{\Re_7}{\Re_6}.$$

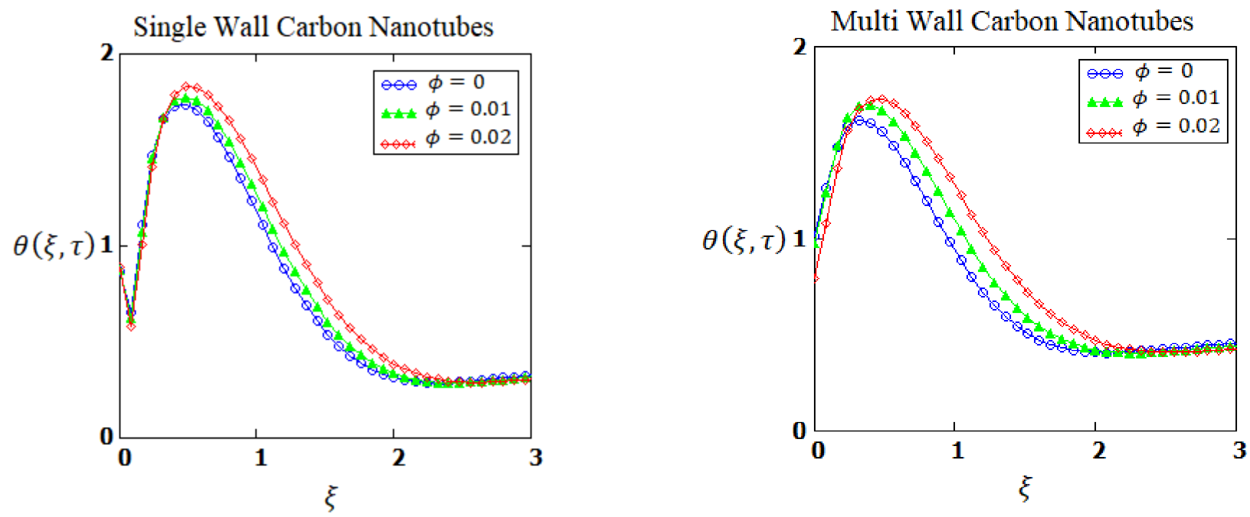

Fig. 2. Plot for SWCNTs versus MWCNTs with varying values of volume fraction on temperature distribution.

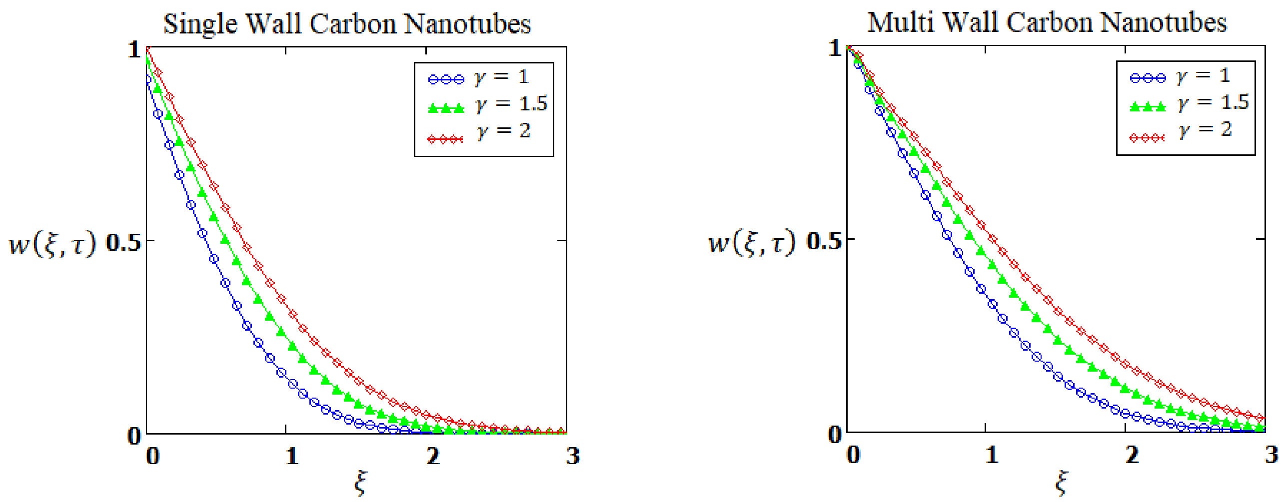

Fig. 3. Plot for SWCNTs versus MWCNTs with varying values of Casson fluid parameter on velocity field.

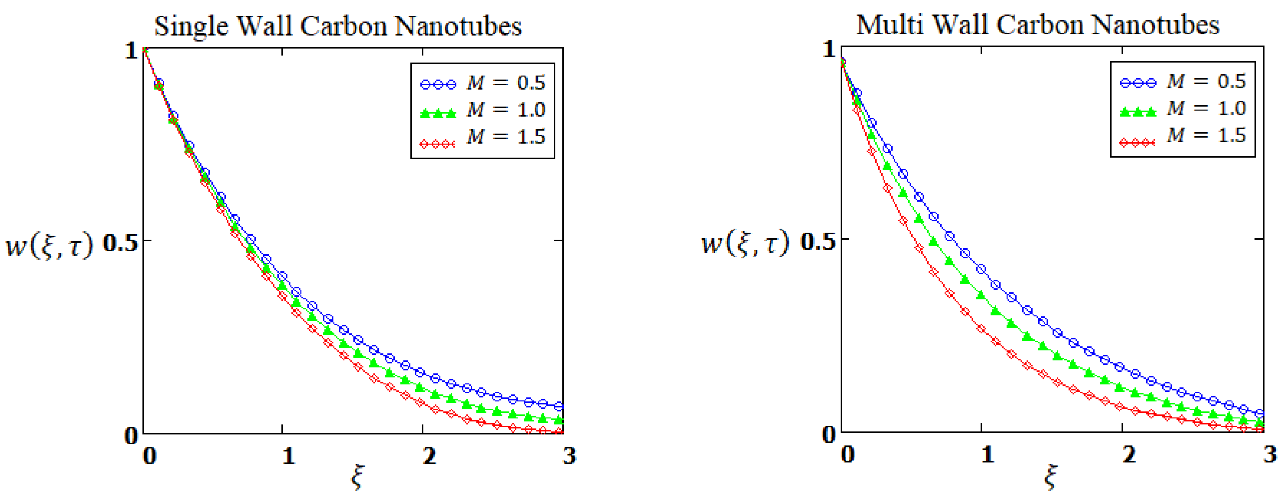

Fig. 4. Plot for SWCNTs versus MWCNTs with varying values of magnetic field on velocity field.

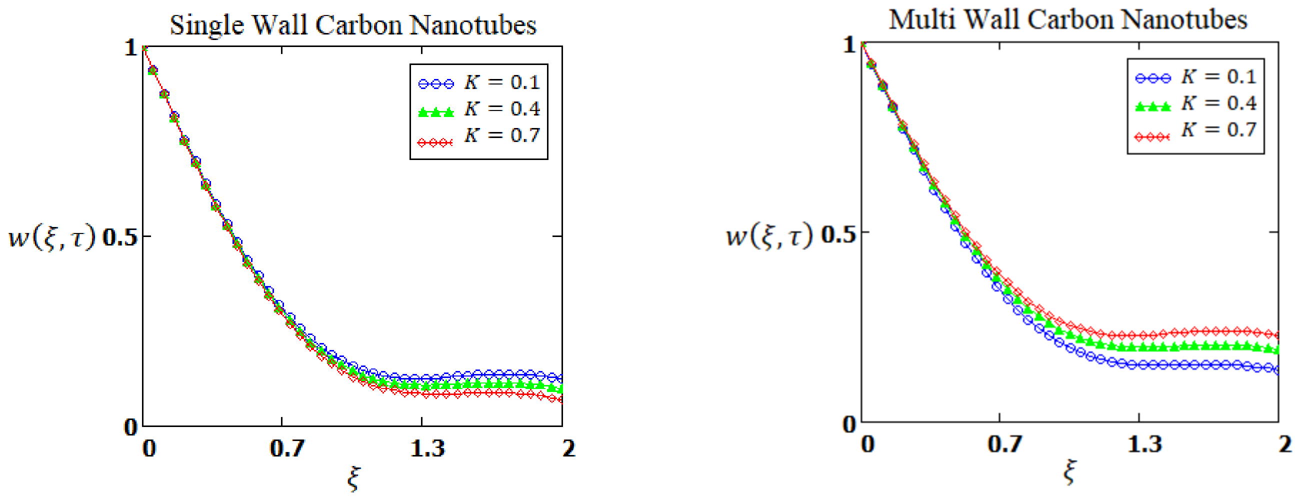

Fig. 5. Plot for SWCNTs versus MWCNTs with varying values of porous medium on velocity field.

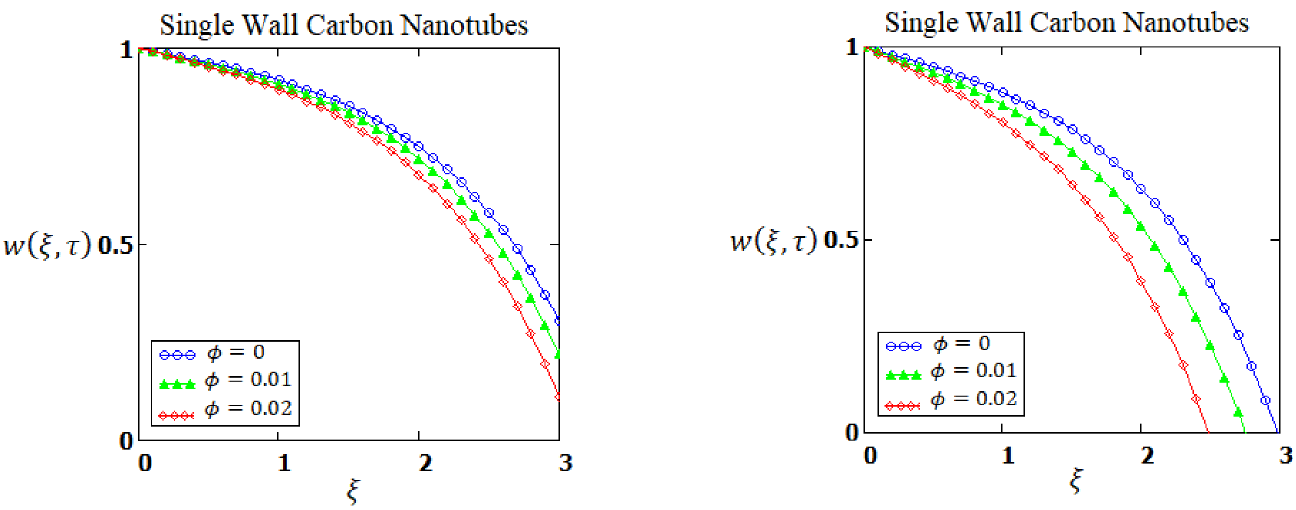

Fig. 6. Plot for SWCNTs versus MWCNTs with varying values of volume fraction on velocity field.

The equivalent expression of equation (24) was investigated by employing suitable generalized series for fulfilling the imposed conditions on governing equations, we recovered

$$\bar{w}(\xi,q) = \frac{p!}{q^{p+1}} + \frac{p!}{q^{p+1}} \sum_{\eta_2=1}^{\infty}\left(-\xi\sqrt{\Re_1^{-1}\beta\left(\frac{1}{1+\gamma}\right)^{-1}}\right)^{\eta_2} \sum_{\eta_3=0}^{\infty}\frac{1}{\eta_3!}\left(\frac{\Re_9}{\beta}\right)^{\eta_3}\sum_{\eta_4=0}^{\infty}\frac{(-\alpha\beta)^{\eta_4}}{m!\,q^{\alpha\eta_4}}$$

$$\times \frac{\Gamma\left(\eta_3-\frac{\eta_2}{2}+1\right)\Gamma\left(\frac{\eta_2}{2}+1\right)}{\Gamma\left(\eta_3-\frac{\eta_2}{2}-\eta_4+1\right)\Gamma\left(\frac{\eta_2}{2}-\eta_3+1\right)} - \frac{Gr\,\Re_5}{\Re_8}\sum_{\eta_2=0}^{\infty} -\xi\left(\sqrt{\lambda^{-1}\Pr\Re_4\beta}\right)^{\eta_2}\sum_{\eta_3=0}^{\infty}\left(\frac{1}{\Re_4}\right)^{\eta_3} \tag{25}$$

$$\times\sum_{\eta_3=0}^{\infty}\frac{(-\alpha\beta)^{\eta_4}}{\eta_4!\,q^{\alpha\eta_4}}\frac{\Gamma\left(\frac{\eta_2}{2}+2\right)}{\Gamma\left(\frac{\eta_2}{2}-\eta_4+2\right)}\frac{1}{q^{\alpha\eta_3+\eta_4}},$$

where $\Re_9 = (\Re_1\Re_2 - Mk)/\Re_2 k$. Expressing equation (25) by means of special function [51] and inverting it via Laplace transform, we obtain

$$w(\xi,\tau) = H(\tau)\tau^{p+1} + \sum_{\eta_2=1}^{\infty}\left(-\xi\sqrt{\Re_1^{-1}\beta\left(\frac{1}{1+\gamma}\right)^{-1}}\right)^{\eta_2}\sum_{\eta_3=0}^{\infty}\frac{1}{\eta_3!}\left(\frac{\Re_9}{\beta}\right)^{\eta_3} \tag{26}$$

$$\times\int_0^{\tau}(\tau-z)^{p}\mathbf{H}_{2,4}^{1,2}\left[-\alpha\beta\tau^{\alpha}\left|\begin{matrix}\left(-\frac{\eta_2}{2},0\right),\left(\frac{\eta_2}{2}-\eta_3,0\right)\\(0,1)\left(\eta_3-\frac{\eta_2}{2},0\right),\left(\frac{\eta_2}{2}-\eta_3,-1\right),(0,\alpha)\end{matrix}\right.\right]dz$$

$$-\frac{Gr\,\Re_5}{\Re_8}\sum_{\eta_2=0}^{\infty}-\xi\left(\sqrt{\lambda^{-1}\Pr\Re_4\beta}\right)^{\eta_2}\sum_{\eta_3=0}^{\infty}\left(\frac{\tau^{\alpha}}{\Re_4}\right)^{\eta_3}\mathbf{H}_{1,3}^{1,1}\left[-\alpha\beta\tau\left|\begin{matrix}\left(-\frac{\eta_2}{2}-1,0\right)\\(0,1)\left(-\frac{\eta_2}{2}-1,-1\right),(-\alpha\eta_3,1)\end{matrix}\right.\right]dz.$$

It is worth pointed out that the equations (21) and (26) are the exact solutions obtained via Atangana-Baleanu fractional operator; both solutions of temperature and velocity distribution satisfy initial and boundary condition as well. The solutions of equations (21) and (26) can be also be particularized for the ordinary partial differential equations by letting $\alpha=1$. Furthermore, the solutions for the velocity field and temperature distribution are achieved by substituting $\gamma=0$ in the equations (21) and (26), such similar solutions have been investigated by [49].

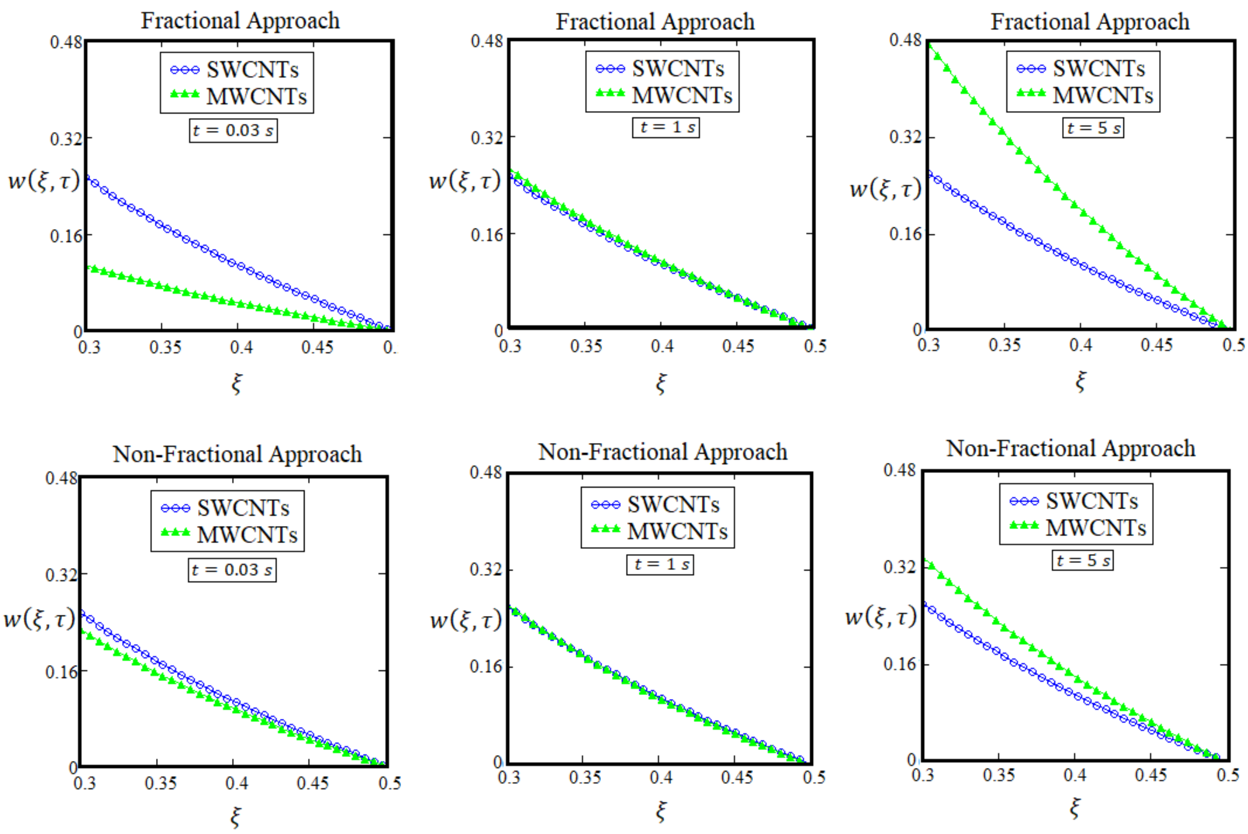

Fig. 7. Plot for SWCNTs verses MWCNTs with fractional and non-fractional (ordinary) approaches on velocity field for three different times.

4. Results and Conclusions

In this article, we analyzed the carbon nanotubes based Casson nanofluid suspended in ethylene glycol by employing modern method of fractional calculus namely Atangana-Baleanu fractional derivative. The governing equations are solved analytically by invoking Laplace transforms. The analytic solutions are established for the temperature and velocity distribution and expressed in terms of special function. The graphical results are depicted and then discussed for carbon nanotubes with various embedded parameters in this portion. Furthermore, the main findings are examined with the contributions of different physical parameters like Casson-nanofluid parameter of Casson-nanofluid γ, solid volume fraction of Casson-nanofluid ϕ, permeability parameter K, magnetic parameter M, Prandtl number $\Pr$, Grashof number Gr, carbon nanotubes and few others.

(i) Fig. 2 is depicted to elucidate the effects of nanoparticles volume fraction on temperature distribution for carbon nanotubes. It is observed that temperature distribution is not decaying function of nanoparticles volume fraction either in single walled carbon nanotubes or in multi walled carbon nanotubes. Physically, temperature distribution of Casson-nanofluid increases the corresponding thermal boundary layer for varying values of nanoparticles volume fraction. This may be due to the fact that the optimum value of any volume fraction depends on the reinforcing material; i-e carbon nanotubes.

(ii) The influence of Casson-fluid parameter on carbon nanotubes is captured in Fig. 3. As expected, increase in Casson fluid parameter increases the velocity field for either single walled carbon nanotubes or multi-walled carbon nanotubes. Physically it is because of nanotubes structure in chiral, when Casson-fluid parameter increases then nanotubes do not cause the deceleration in velocity field.

(iii) The magnetic findings of the carbon nanotubes are investigated in Fig. 4. As the carbon nanotubes have various patterns of carbon, for this reason, magnetic field has very interesting result on carbon nanotubes. Enhancing the magnetic value results the reduction in the velocity boundary layer thickness for single walled carbon nanotubes, this may be due to the fact of the transport properties of carbon nanotubes. On the contrary, the magnetic field contact can magnetically polarize the nanotubes; this is why the similar trend is observed for multi-walled carbon nanotubes.

(iv) Porous carbon nanotubes are depicted in Fig. 5 on the mobility of carbon nanotubes. The presence of porous medium in both types of carbon nanotubes has opposite trend which is perceived for the resistance of fluid velocity.

(v) Fig. 6 is depicted for the significance of the nanoparticles volume fraction on velocity field between the range $\phi = 0, 0.01, 0.02$ for carbon nanotubes. It is observed that both types of carbon nanotubes have similar behavior on account of velocity field. In simplicity, increase of nanoparticle volume fraction has no major effect for carbon nanotubes.

(vi) Fig. 7 depicts the visualizations of carbon nanotubes with fractional and non-fractional approaches at three different times $t = 0.03s, t = 1s, t = 5s$. For smaller time $t = 0.03s$, single walled carbon nanotube is accelerated either in fractional approach or non-fractional approach; for larger time $t = 5s$, single walled carbon nanotube is decelerated but multi-walled carbon nanotube is rapid in comparison with single walled carbon nanotube either in fractional or non-fractional approaches. On the other hand, for unit time $t = 1s$, single and multi-walled carbon nanotubes have identical velocities with and without fractional approach. The same dynamics of carbon nanotubes can also be signified for temperature analysis. In brevity, the rheological characteristics of both types of carbon nanotubes can also be analyzed when carbon nanotubes are suspended in ethylene glycol as well.

Author Contributions

K.A. Abro, M.H. Laghari and J.F. Gómez-Aguilar developed the mathematical modeling and examined the theory validation. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

Acknowledgments

José Francisco Gómez Aguilar acknowledges the support provided by CONACyT: Cátedras CONACyT para jóvenes investigadores 2014 and SNI CONACyT.

Conflict of Interest

There is no competing interest among the authors regarding the publication of the article.

Funding

The authors received no financial support for the research, authorship and publication of this article.

References

[1] Choi, S.U.S., Zhang, Z.G., Yu, W., Lockwood, F.E., Grulke, E.A., Anomalous Thermal Conductivity Enhancement in Nanotube Suspensions. *Applied Physics Letters*, 79(14), 2001, 2252-2254.

[2] Ramasubramaniam, R., Chen, J., Liu, H., Homogeneous Carbon Nanotube/Polymer Composites for Electrical Applications. *Applied Physics Letters*, 83, 2003, 2928-2930.

[3] Xue, Q.Z., Model for Thermal Conductivity of Carbon Nanotube-Based Composites. *Physica B*, 7, 2005, 302-307.

[4] Berber, S., Kwon, Y.K., Tomanek, D., Unusually High Thermal Conductivity of Carbon Nanotubes, *Physical Review Letters*, 84(20), 2000, 4613-4616.

[5] Hone, J., Llaguno, M.C., Biercuk, M.J., Johnson, A.T., Batlogg, B., Benes, Z., Fischer, J.E., Thermal Properties of Carbon Nanotubes and Nanotube-Based Material. *Applied Physics A*, 74, 2002, 339–343.

[6] Jiang, W., Ding, G., Pang, H., Measurement and Model on Thermal Conductivities of Carbon Nanotube Nano-Refrigerants. *International Journal of Thermal Sciences*, 48, 2009, 1108–1115.

[7] Tan, G., Mieno, T., Experimental and Numerical Studies of Heat Convection in the Synthesis of Single-Walled Carbon Nanotubes by Arc Vaporization. *Japanese Journal of Applied Physics*, 49(4), 2010, 1-9.

[8] Mayer, J., Mckrell, T., Grote, K., The Influence of Multi-Walled Carbon Nanotubes on Single-Phase Heat Transfer and Pressure Drop Characteristics in the Transitional Flow Regime of Smooth Tubes. *International Journal of Heat and Mass Transfer*, 58(1-2), 2013, 597–609.

[9] Haq, R.U., Hammouch, Z., Khan, W.A., Water-Based Squeezing Flow in the Presence of Carbon Nanotubes Between Two Parallel Disks, *Thermal Sciences*, 1, 2014, 148–158.

[10] Haq, R.U., Khan, Z.H., Khan, W.A., Thermophysical Effects of Carbon Nanotubes on MHD Flow Over a Stretching Surface. *Physica E*, 63, 2014, 215–222.

[11] Ellahi, R., Hassan, M., Zeeshan, A., Study of Natural Convection MHD Nanofluid by Means of Single and Multi Walled Carbon Nanotubes Suspended in a Salt Water Solutions. *IEEE Transactions on Nanotechnology*, 14, 2015, 726–734.

[12] Khan, U., Ahmed, N., Tauseef Mohyud-Din, S., Numerical Investigation for Three Dimensional Squeezing Flow of Nanofluid in a Rotating Channel with Lower Stretching Wall Suspended by Carbon Nanotubes. *Applied Thermal Engineering*, 1, 2016, 1-11.

[13] Hayat, T., Khan, M.I., Farooq, M., Alsaedi, A., Yasmeen, T., Impact of Marangoni Convection in the Flow of Carbon-Water Nanofluid with Thermal Radiation. *International Journal of Heat and Mass Transfer*, 106, 2017, 810-815.

[14] Khan, U., Ahmed, N., Mohyud-Din, S.T., Stoke's First Problem for Carbon Nanotubes Suspended Nanofluid Flow Under the Effect of Slip Boundary Condition. *Journal of Nanofluids*, 5, 2016, 239-244.

[15] Khan, W.A., Khan, Z.H., Rahi. M., Fluid Flow and Heat Transfer of Carbon Nanotubes Along a Flat Plate with Navier Slip Boundary. *Applied Nanoscience*, 4(5), 2014, 633-641.

[16] Fleming, E., Du, F., Ou, E., Dai, L., Shi. L., Thermal Conductivity of Carbon Nanotubes Grown by Catalyst-Free Chemical Vapor Deposition in Nanopores. *Carbon*, 145, 2019, 195-200.

[17] Ghazanfari, S.A., Wahid, M.A., HEAT TRANSFER ENHANCEMENT AND PRESSURE DROP FOR FIN-AND-TUBE COMPACT HEAT

EXCHANGERS WITH DELTA WINGLET-TYPE VORTEX GENERATORS, *Facta Universitatis, Series: Mechanical Engineering*, 16(2), 2018, 233-247.
[18] Bhojraj, L., Kashif, A.A., Abdul, W.S., Thermodynamical Analysis of Heat Transfer of Gravity-Driven Fluid Flow via Fractional Treatment: An Analytical Study. *Journal of Thermal Analysis and Calorimetry*, (2020) https://doi.org/10.1007/s10973-020-09429-w.
[19] Acar, B., Laminar Forced Convection of Various Nanofluids in Sudden Expansion Channels Under Constant Heat Flux: A CFD Study, *International Journal of Applied Mechanics*, 11(5), 2019, 1950049.
[20] Farahi Shahri, m., Hossein Nezhad, A., Application of Various Electromagnetic Coupling Modes for the Better MHD Flow Distribution and Thermal Management Within a Liquid Metal Manifold, *International Journal of Applied Mechanics*, 10(5), 2018, 1850052.
[21] Aziz, U.A., Mukarram, A., Kashif, A.A., Electroosmotic Slip Flow of Oldroyd-B Fluid Between Two Plates with Non-Singular Kernel. *Journal of Computational and Applied Mathematics*, 376, 2020, 112885.
[22] Dogonchi, A.S., Chamkha, A.J., Seyyedi, S.M., Hashemi-Tilehnoee, M., Ganji, D.D., Viscous Dissipation Impact on Free Convection Flow of Cu-Water Nanofluid in a Circular Enclosure with Porosity Considering Internal Heat Source. *Journal of Applied and Computational Mechanics*, 5(4), 2019, 717-726.
[23] Koca, I., Atangana, A., Solutions of Cattaneo-Hristov Model of Elastic Heat Diffusion with Caputo-Fabrizio and Atangana-Baleanu Fractional Derivatives. *Thermal Science*, 21(6A), 2017, 2299-2305.
[24] Abro, K.A., Abro, I.A., Almani, S.M., Khan, I., On the Thermal Analysis of Magnetohydrodynamic Jeffery Fluid via Modern Non Integer Order derivative. *Journal of King Saud University – Science*, 31(4), 2019, 973-979.
[25] Sheikholeslami, M., Ellahi, R., Three Dimensional Mesoscopic Simulation of Magnetic Field Effect on Natural Convection of Nanofluid. International Journal of Heat and Mass Transfer, 89, 2015, 799–808.
[26] Abro, K.A., Rashidi, M.M., Khan, I., Abro, I.A., Tassadiq, A., Analysis of Stokes' Second Problem for Nanofluids Using Modern Fractional Derivatives. *Journal of Nanofluids*, 7, 2018, 738–747.
[27] Sheikholeslami, M., Ellahi, R., Simulation of Ferrofluid Flow for Magnetic Drug Targeting Using Lattice Boltzmann Method. *Zeitschrift Fur Naturforschung A*, 70, 2015, 115–124.
[28] Khan, I., Abro, K.A., Thermal Analysis in Stokes' Second Problem of Nanofluid: Applications in Thermal Engineering, *Case Studies in Thermal Engineering*, 12, 2018, 271-275.
[29] Zeehan, A., Ellahi, R., Hassan, M., Magnetohydrodynamic Flow of Water/Ethylene Glycol Based Nanofluids with Natural Convection Through Porous Medium. *European Physical Journal Plus*, 129, 2014, 1-12.
[30] Ellahi, R., Aziz, S., Zeeshan, A., Non-Newtonian Nanofluids Flow Through a Porous Medium Between Two Coaxial Cylinders with Heat Transfer and Variable Viscosity. *Journal of Porous Media*, 16(3), 2013, 205–216.
[31] Abro, K.A., Chandio, A.D., Abro, I.A., Khan, I., Dual Thermal Analysis of Magnetohydrodynamic Flow of Nanofluids via Modern Approaches of Caputo–Fabrizio and Atangana–Baleanu Fractional Derivatives Embedded in Porous Medium. *Journal of Thermal Analysis and Calorimetry*, 135, 2019, 2197–2207.
[32] Rehman, S.U., Haq, R.U., Khan, Z.H., Lee, C., Entropy Generation Analysis for Non-Newtonian Nanofluid with Zero Normal Flux of Nanoparticles at the Stretching Surface. *Journal of the Taiwan Institute of Chemical Engineers*, 63, 2016, 226-235.
[33] Khan, Z.H., Hussain, S.T., Hammouch, Z., Flow and Heat Transfer Analysis of Water and Ethylene Glycol Based Cu nanoparticles between two parallel disks with suction/injection effects. *Journal of Molecular Liquids*, 221, 2016, 298-304.
[34] Kashif, A.A, Abdon, A., Role of Non-integer and Integer Order Differentiations on the Relaxation Phenomena of Viscoelastic Fluid, *Physica Scripta*, 2020, doi: 10.1088/1402-4896/ab560c
[35] Atangana, A., Koca, I., On the New Fractional Derivative and Application to Nonlinear Baggs and Freedman Model. *Journal of Nonlinear Sciences and Applications*, 9, 2016, 2467-2480.
[36] Abro, K.A., Gomez-Aguilar, J.F., Khan, I., Nisar, K.S., Role of Modern Fractional Derivatives in an Armature-Controlled DC Servomotor. *European Physical Journal Plus*, 134(553), 2019, 1-16.
[37] Yavuz, M., Özdemir, N., Comparing the New Fractional Derivative Operators Involving Exponential and Mittag-Leffler Kernel. *Discrete & Continuous Dynamical Systems-S*, 2, 2019, 1098-1107.
[38] Atangana, A., Baleanu, D., New Fractional Derivatives with Nonlocal and Nonsingular Kernel: Theory and Application to Heat Transfer Model. *Thermal Science*, 20(2), 2016, 763-769.
[39] Abro, K.A., Pervaiz, H.S., Jose, F.G.A, Ilyas, K., Analysis of De-Levie's Model via Modern Fractional Differentiations: An Application to Supercapacitor. *Alexandria Engineering Journal*, 58(4), 2019 58, 1375–1384.
[40] Abro, K.A., Hussain, M., Baig, M.M., An Analytic Study of Molybdenum Disulfide Nanofluids Using Modern Approach of Atangana-Baleanu Fractional Derivatives. *European Physical Journal Plus*, 132, 2017, 1-14.
[41] Atangana, A., Baleanu, D., Caputo–Fabrizio Derivative Applied to Groundwater Flow Within Confined Aquifer. *Journal of Engineering Mechanics*, 142, 2016, 1-8.
[42] Abro, K.A., Irfan, A.A, Ahmed, Y., A Comparative Analysis of Sulfate Ion Concentration via Modern Fractional Derivatives: An Industrial Application to Cooling System of Power Plant. *Physica A: Statistical Mechanics and its Applications*, 541, 2020, 123306.
[43] Abro, K.A., Memon, A.A., Uqaili, M.A., A Comparative Mathematical Analysis of RL and RC Electrical Circuits via Atangana-Baleanu and Caputo-Fabrizio Fractional Derivatives. *European Physical Journal Plus*, 133, 2018, 1-8.
[44] Kashif, A.A., Atangana, A., Mathematical Analysis of Memristor Through Fractal-Fractional Differential Operators: A Numerical Study, *Mathematical Methods in the Applied Sciences*, 2020, https://doi.org/10.1002/mma.6378.
[45] Casson, N., *A Flow Equation for the Pigment Oil Suspensions of the Printing Ink Type in Rheology of Disperse Systems*. Pergamon, New York, NY, USA, 1, 1959.
[46] Wang, J., Zhu, J., Zhang, X., Chen, Y., Heat Transfer and Pressure Drop of Nanofluids Containing Carbon Nanotubes in Laminar Flows. *Experimental Thermal and Fluid Science,* 44, 2013, 716-721.
[47] Mustafa, M., Khan, J.A., Model for Flow of Casson Nanofluid Past a Nonlinearly Stretching Sheet Considering Magnetic Field Effects. *AIP Advances*, 5, 2015, 1-9.
[48] Abro, K.A., Khan, I., Analysis of Heat and Mass Transfer in MHD Flow of Generalized Casson Fluid in a Porous Space Via Non-Integer Order Derivative without Singular Kernel. *Chinese Journal of Physics*, 55(4), 2017, 1583-1595.
[49] Chaudhary, R.C., Jain, P., Unsteady Free Convection Boundary Layer Flow Past an Impulsively Started Vertical Surface with Newtonian Heating. *Romanian Journal of Physics*, 51, 2006, 911–925.
[50] Al-Mdallal, Q., Abro, K.A., Khan, I., Analytical Solutions of Fractional Walter's-B Fluid with Applications. *Complexity*, 1, 2018, 1-8.
[51] Abro, K.A., Solangi, M.A., Heat Transfer in Magnetohydrodynamic Second Grade Fluid with Porous Impacts using Caputo-Fabrizio Fractional Derivatives. *Punjab University Journal of Mathematics*, 49(2), 2017, 113-125.
[52] Ouakad, H.M., Sedighi, H.M., Rippling Effect on the Structural Response of Electrostatically Actuated Single-Walled Carbon Nanotube Based NEMS Actuators. *International Journal of Non-Linear Mechanics*, 87, 2016, 97-108.

[53] Sedighi, H.M., Daneshmand, F., Static and Dynamic Pull-in Instability of Multi-Walled Carbon Nanotube Probes by He's Iteration Perturbation Method. *Journal of Mechanical Science and Technology*, 28(9), 2014, 3459-3469.
[54] Abro, K.A., Atangana, A., A Comparative Study of Convective Fluid Motion in Rotating Cavity via Atangana–Baleanu and Caputo–Fabrizio Fractal-Fractional Differentiations. European Physical Journal Plus, 135(226), 2020, 1-12.
[55] Kashif, A.A., Siyal, A., Atangana, A., Thermal Stratification of Rotational Second-Grade Fluid through Fractional Differential Operators. *Journal of Thermal Analysis and Calorimetry*, 2020, https://doi.org/10.1007/s10973-020-09312-8.
[56] Eltaher, M., Agwa, M., Kabeel, A., Vibration Analysis of Material Size-Dependent CNTs Using Energy Equivalent Model, *Journal of Applied and Computational Mechanics*, 4(2), 2018, 75-86.
[57] Sedighi, H.M., Yaghootian, A., Dynamic Instability of Vibrating Carbon Nanotubes near Small Layers of Graphite Sheets Based on Nonlocal Continuum Elasticity. *Journal of Applied Mechanics and Technical Physics*, 57(1), 2016, 90-100.
[58] Sedighi, H.M., Farjam, N., A Modified Model for Dynamic Instability of CNT Based Actuators by Considering Rippling Deformation Tip-Charge Concentration and Casimir Attraction. *Microsystem Technologies*, 23(6), 2016, 2175-2191.
[59] Kashif, A.A., A Fractional and Analytic Investigation of Thermo-Diffusion Process on Free Convection Flow: An Application to Surface Modification Technology. *European Physical Journal Plus*, 1, 2020, 1-16.
[60] Sedighi, H.M., Malikan, M., Stress-Driven Nonlocal Elasticity for Nonlinear Vibration Characteristics of Carbon/Boron-Nitride Hetero-Nanotube Subject to Magneto-Thermal Environment. *Physica Scripta*, 95(5), 2020, 055218.
[61] Sedighi, H.M., Divergence and Flutter Instability of Magneto-Thermo-Elastic C-BN Hetero-Nanotubes Conveying Fluid. *Acta Mechanica Sinica*, 36, 2020, 381–396.

ORCID iD

Kashif Ali Abro https://orcid.org/0000-0003-0867-642X
Muzaffar Hussain Laghari https://orcid.org/0000-0003-4568-9537
J.F. Gómez-Aguilar https://orcid.org/0000-0001-9403-3767

How to cite this article: Abro K.A., Laghari M.H., Gómez-Aguilar J.F. Application of Atangana-Baleanu Fractional Derivative to Carbon Nanotubes Based Non-Newtonian Nanofluid: Applications in Nanotechnology, *J. Appl. Comput. Mech.*, 6(SI), 2020, 1260–1269. https://doi.org/10.22055/JACM.2020.33461.2229