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Research Paper

Modified Variational Iteration Technique for the Numerical Solution of Fifth Order KdV-type Equations

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Abstract. In this article, a simple and new algorithm is proposed, namely the modified variational iteration algorithm-I (mVIA-I), for obtaining numerical solutions to different types of fifth-order Korteweg de-Vries (KdV) equations. In order to verify the precision, accuracy and stability of the mVIA-I method, generated numerical results are compared with the Laplace decomposition method, Adomian decomposition method, Homotopy perturbation transform method and the modified Adomian decomposition method. Comparison with the mentioned methods reveals that the mVIA-I is computationally attractive, exceptionally productive and achieves better accuracy than the others.

Keywords: Korteweg–de Vries equation, Modification of variational iteration algorithm-I, Fifth order KdV equation, Generalized KdV equation.

1. Introduction

Nonlinear phenomena appear in many areas of scientific and engineering fields, for example, fluid dynamics, quantum mechanics, nonlinear optics, plasma physics, chemical kinetics, solid-state physics and mathematical biology and so forth. All these phenomena are modeled as nonlinear partial differential equations (PDEs) [1]–[10]. Among these PDEs, the generalized fifth order Korteweg-de Vries (gfKdV) equations are used to study various important topics in nonlinear physical phenomena. This equation not just portrays the movement of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice, but at the same time, it is a significant mathematical model for magneto-sound propagation in plasmas [11] and a chain of coupled nonlinear oscillators [12]. The gfKdV equation is very difficult and still, general exact solutions of these highly nonlinear physical problems are not known. So far, the exact solution of the gfKdV equation is found for the special case of solitary waves in [13]. In general, many analytical and numerical approaches are used for finding the solution to the gfKdV equation. The steady solution of the fifth-order Korteweg de-Vries (KdV) equation is investigated in [11], while the proposed finite difference method is employed for solving this equation in [14]. In [15], first reduced this equation to ODEs and then developed several analytical and numerical methods. However, these strategies are difficult to utilize and some of the time requires repetitive work and calculation [16]. Recently, the decomposition method is used for the numerical solutions of the fifth-order KdV equation in [17] whereas modified Adomian decomposition method is used in [18]. Also, He's semi inverse scheme [19], Laplace decomposition approach [20], Differential transform method [21], the tanh and sine–cosine methods [22], Homotopy analysis technique [23], Laplace decomposition method [24], finite difference schemes [14], homotopy perturbation method [25], Fractional homotopy analysis transform technique [26], Exp-function method [27], Variational iteration technique [28]–[31], homogeneous balance method and many more methods can be found in the literature for solving such types of highly nonlinear physical problems.

The KdV equation is a nonlinear partial differential equation which is extensively arising in various physical applications and plays a very significant role in numerous applications like in shallow water waves in plasmas, nonlinear LC circuit's waves, ion-acoustic waves and magneto-acoustic waves. This equation has experienced a few modifications and extensions leading to several variants being presented up in three, five, seven or higher-order differential equations. Though, some noteworthy types of them are specifically significant in modeling physical phenomena. The main types of the KdV equation of fifth order which are the indispensable models for numerous physical phenomena

1. The Sawada-Kotera (SK) equation [32]

$$\frac{\partial w}{\partial t} + 45\, w^2 \frac{\partial w}{\partial x} + 15 \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + 15\, \frac{\partial^3 w}{\partial x^3} + \frac{\partial^5 w}{\partial x^5} = 0. \tag{1}$$

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2. The Caudrey-Dodd-Gibbon equation [33]

$$\frac{\partial w}{\partial t}+180\,w^2\frac{\partial w}{\partial x}+30\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+30\,\frac{\partial^3 w}{\partial x^3}+\frac{\partial^5 w}{\partial x^5}=0. \tag{2}$$

3. The Kaup-Kuperschmidt equation [33]

$$\frac{\partial w}{\partial t}+20\,w^2\frac{\partial w}{\partial x}+25\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+10\,\frac{\partial^3 w}{\partial x^3}+\frac{\partial^5 w}{\partial x^5}=0. \tag{3}$$

4. The Lax equation [34]

$$\frac{\partial w}{\partial t}+30\,w^2\frac{\partial w}{\partial x}+30\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+10\,\frac{\partial^3 w}{\partial x^3}+\frac{\partial^5 w}{\partial x^5}=0. \tag{4}$$

5. The Ito equation [33]

$$\frac{\partial w}{\partial t}+2\,w^2\frac{\partial w}{\partial x}+6\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+3\,\frac{\partial^3 w}{\partial x^3}+\frac{\partial^5 w}{\partial x^5}=0. \tag{5}$$

6. Fifth order KdV equation [24]

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}-w\frac{\partial^3 w}{\partial x^3}+\frac{\partial^5 w}{\partial x^5}=0, \tag{6}$$

7. Kawahara equation [35]

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}+\frac{\partial^3 w}{\partial x^3}-\frac{\partial^5 w}{\partial x^5}=0. \tag{7}$$

The main aim of this study is to use mVIA-I which is the modified form of variational iteration method [36] for finding the numerical solution of nonlinear PDEs in physical sciences and engineering which are modeled via the fifth order KdV equation. The paper is organized in the following sections. In section 1, the proposed method and its implementation are described. In section 2, seven different types of KdV equations are numerically solved to show the applicability and accuracy of the modified algorithm. Some concluding remarks are discussed in the last section 3.

2. Implementation of mVIA-I for Fifth Order PDEs

In this section, we illustrate mVIA-I for the numerical treatment of different KdV type equations of the fifth order. Consider the generalized form of the fifth order KdV equation

$$\frac{\partial w}{\partial t}+\alpha\,w\frac{\partial^3 w}{\partial x^3}+\beta\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\gamma\,w^2\frac{\partial w}{\partial x}+\frac{\partial^5 w}{\partial x^5}=0, \tag{8}$$

where γ, β and α are nonzero arbitrary constants.
Approximate solution $w_{k+1}(x)$ of the eq. (8) for given initial condition $w_0(x)$ can be acquired as below

$$w_{k+1}(x,t,h)=w_0(x,t,h)+h\int_0^t\lambda(\eta)\left[\frac{\partial w_k(x,\eta,h)}{\partial(\eta)}+\overbrace{\alpha\,w_k(x,\eta,h)\frac{\partial^3 w_k(x,\eta,h)}{\partial(x)^3}}+\beta\overbrace{\frac{\partial w_k(x,\eta,h)}{\partial(x)}\frac{\partial^2 w_k(x,\eta,h)}{\partial(x)^2}}+\overbrace{\gamma\,w_k^2(x,\eta,h)\frac{\partial w_k(x,\eta,h)}{\partial(x)}}+\overbrace{\frac{\partial^5 w_k(x,\eta,h)}{\partial(x)^5}}\right]d\eta, \tag{9}$$

Where λ and h are unknown parameters. The first one is known as the Lagrange multiplier [37], while the second one is an auxiliary parameter, which was used to accelerate the convergence in different methods [38]–[44].
The significant value of the Lagrange multiplier can be achieved either by variational principle reported in [45]–[47] or by applying δ on both sides of the recurrent relation (9) with respect to $w_k(x)$, which leads to

$$\begin{aligned}\delta w_{k+1}(x,t,h)&=\delta w_0(x,t,h)\\&+h\delta\int_0^t\lambda(\eta)\left[\frac{\partial w_k(x,\eta,h)}{\partial(\eta)}+\overbrace{\alpha\,w_k(x,\eta,h)\frac{\partial^3 w_k(x,\eta,h)}{\partial(x)^3}}+\beta\overbrace{\frac{\partial w_k(x,\eta,h)}{\partial(x)}\frac{\partial^2 w_k(x,\eta,h)}{\partial(x)^2}}+\overbrace{\gamma\,w_k^2(x,\eta,h)\frac{\partial w_k(x,\eta,h)}{\partial(x)}}\right.\\&\left.+\overbrace{\frac{\partial^5 w_k(x,\eta,h)}{\partial(x)^5}}\right]d\eta,\end{aligned} \tag{10}$$

where $\overbrace{w_k(x,\eta,h)}$ is treated as a restricted term, such that, $\delta\overbrace{w_k(x,\eta,h)}=0$. The constant h is an auxiliary parameter which is used to make sure the convergence of approximated solutions to the exact one by limiting the norm 2 of residual error over the space of the given system.

The optimal value of h improves the proficiency and precision of the algorithm. For this purpose, a residual function for approximate solution is defined as

$$r_p(x,\,t,\,h)=\frac{\partial w_p(x,\,t,\,h)}{\partial(t)}+\alpha w_p(x,\,t,\,h)\frac{\partial^3 w_p(x,\,t,\,h)}{\partial(x)^3}+\beta\frac{\partial w_p(x,\,t,\,h)}{\partial(x)}\frac{\partial^2 w_p(x,\,t,\,h)}{\partial(x)^2}\gamma\,w_p^2(x,t,h)\frac{\partial w_p(x,\,t,\,h)}{\partial(x)}+\frac{\partial^5 w_p(x,\,t,\,h)}{\partial(x)^5}, \tag{11}$$

where the number of approximations is denoted by p. The square of the function (11) for the p[th]-order approximation with respect to the parameter h for $(x,t)\epsilon\,[a,\,b]\times[a,\,b]$ is

$$e_p(\,h)=\left[\frac{1}{(b+1)^2}\sum_{i=a}^{b}\sum_{j=a}^{b}\{r_p(i,j,h)\}^2\right]^{\frac{1}{2}}. \tag{12}$$

The optimal value of h will be chosen at a point where the function (12) gives minimum value. Furthermore, this approach will give the value of h as 1 in small domains, where the standard VIA-I gives high order accuracy.
The recurrence relation gives the following iterative formula after using values of both the parameters:

$$\begin{aligned}\delta w_{k+1}(x,t,h)&=\delta w_0(x,t,h)\\&+h\delta\int_0^t\lambda(\eta)\left[\frac{\partial w_k(x,\eta,h)}{\partial(\eta)}+\alpha\,w_k(x,\eta,h)\frac{\partial^3 w_k(x,\eta,h)}{\partial(x)^3}+\beta\frac{\partial w_k(x,\eta,h)}{\partial(x)}\frac{\partial^2 w_k(x,\eta,h)}{\partial(x)^2}+\gamma\,w_k^2(x,\eta,h)\frac{\partial w_k(x,\eta,h)}{\partial(x)}\right.\\&\left.+\frac{\partial^5 w_k(x,\eta,h)}{\partial(x)^5}\right]d\eta.\end{aligned} \tag{13}$$

Taking start with an appropriate initial guess, the other successive iterative solutions can be obtained by employing the iterative formula (13) and at last, the exact solution $w(x)$ can be obtained as

$$w(x)=\lim_{k\to\infty}w_k(x). \tag{14}$$

This step by step process is termed as mVIA-I. We use this procedure for the numerical treatment of the different types fifth order KdV equations, which is able to give numerical solutions in a direct way and very accurately for linear/nonlinear problems arise in different areas of science and engineering.

3. Numerical Examples

In his section, mVIA-I is used for finding the numerical solution to the Sawada-Kotera (SK) equation, Caudrey-Dodd-Gibbon equation, Kaup-Kuperschmidt equation, a fifth order KdV equation, Lax equation, Ito equation and Kawahara equation. Numerical and graphical results obtained by the proposed process are simulative, constructive, noteworthy, accurate and significant. Illustrated test problems discovered the power and effectiveness of the proposed technique.

3.1. Test Problem 1

Consider the Sawada-Kotera (SK) eq. (1) having the initial condition

$$w(x,0)=2k^2\,\mathrm{sech}^2[k(x-c)], \tag{15}$$

where $sech(z)$ denotes the hyperbolic secant of z.
The exact solution to the SK eq. (1) with the initial condition (15) is given in [18] as

$$w(x,t)=2k^2\,\mathrm{sech}^2[k(x-16k^4t-c)]. \tag{16}$$

The numerical solutions for the test problem 3.1, corresponding to the eq. (1) and generated using mVIA-I, are reported in Table 1. To show the efficiency and applicability of the suggested algorithm in comparison with the Adomian decomposition method (ADM) [48] and the modified ADM [9], the absolute errors are reported in Tables 1, 2 for various values of t and x and for $k=0.01$, and $c=0.0$.
A full agreement between the results of mVIA-I and exact solution can be observed, which confirms the validity of the proposed algorithm. In comparison with the results from [18][48], one can ensure that the results of mVIA-I are more accurate and reliable. Results are also shown graphically, the behavior of approximate and exact solutions can be seen in Fig. 1 for $t=10$ and $k=0.01$ while the absolute error graph and comparison of approximate and exact solutions are shown in Fig. 2 for $k=0.01$.

3.2. Test Problem 2

The Caudrey-Dodd-Gibbon eq. (2) has the initial condition

$$w(x,0)=\frac{k^2e^{kx}}{(1+e^{kx})^2}, \tag{17}$$

and its exact solution was given in [18], as follows

$$w(x,t)=\frac{k^2e^{k(x-k^4t)}}{(1+e^{k(x-k^4t)})^2}. \tag{18}$$

Table 1. Numerical results to the SK equation (1) in terms of absolute errors using mVIA-I for the Test Problem 3.1.

x/t	t=0.2		t=0.4		t=5.0	
	mVIA-I	ADM [18]	mVIA-I	ADM [18]	mVIA-I	ADM [18]
2	6.23416×10^{-19}	1.54499×10^{-18}	7.04731×10^{-19}	4.52654×10^{-18}	7.04731×10^{-19}	6.64350×10^{-15}
4	4.33680×10^{-19}	5.36680×10^{-18}	4.60785×10^{-19}	1.12757×10^{-17}	4.33680×10^{-19}	1.32874×10^{-14}
6	1.89735×10^{-19}	1.13841×10^{-17}	1.62630×10^{-19}	2.02746×10^{-17}	1.35525×10^{-19}	1.99336×10^{-14}
8	7.86046×10^{-19}	1.97054×10^{-17}	8.40256×10^{-19}	3.15232×10^{-17}	7.58941×10^{-19}	2.65820×10^{-14}
10	5.69206×10^{-19}	3.02492×10^{-17}	5.42101×10^{-19}	4.50486×10^{-17}	5.42101×10^{-19}	3.32327×10^{-14}

Table 2. Comparison of numerical solutions to the SK equation (1) for the Test Problem 3.1.

x/t	t=0.2		t=0.4		t=0.5	
	mVIA-I	Modified ADM [48]	mVIA-I	Modified ADM [48]	mVIA-I	Modified ADM [48]
0.1	8.13151×10^{-19}	9.59980×10^{-16}	8.13151×10^{-19}	1.91996×10^{-15}	9.75781×10^{-19}	6.64350×10^{-15}
0.2	8.13151×10^{-20}	1.91996×10^{-15}	2.71050×10^{-20}	3.83989×10^{-14}	2.16840×10^{-19}	1.32874×10^{-14}
0.3	4.87890×10^{-19}	2.87980×10^{-15}	5.96311×10^{-19}	5.75966×10^{-14}	4.06575×10^{-19}	1.99336×10^{-14}
0.5	4.06575×10^{-19}	4.79941×10^{-15}	3.25260×10^{-19}	9.59871×10^{-14}	3.25260×10^{-19}	3.32327×10^{-14}

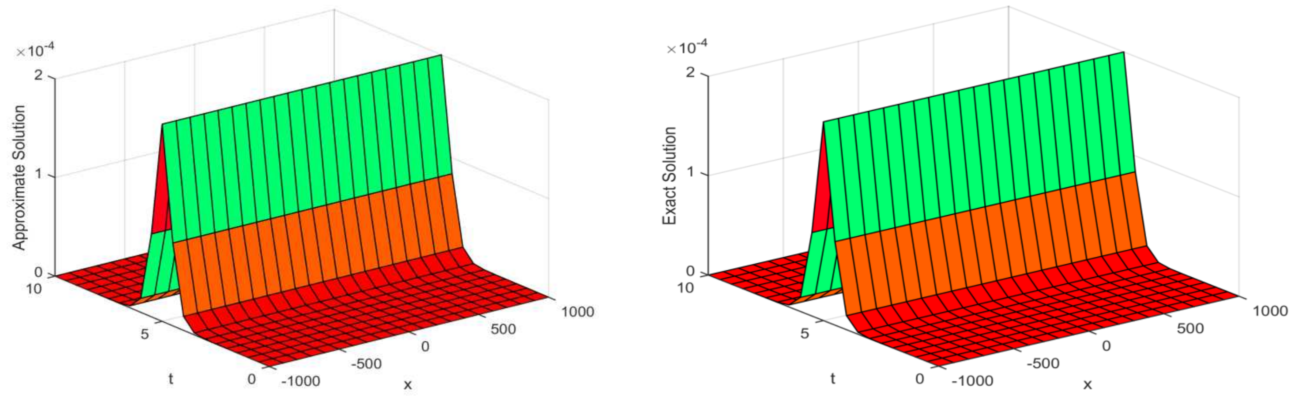

Fig. 1. The numerical solution (above) and exact solution (below) of the Test Problem 3.1 for t = 10 and k = 0.01.

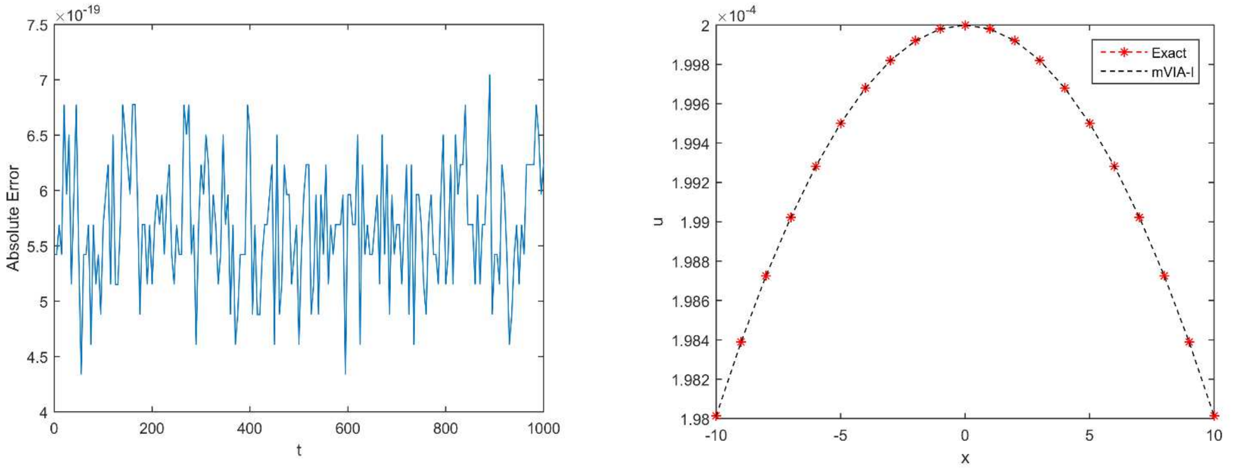

Fig. 2. Absolute error graph for x=10 (left) and comparison of exact and approximate solutions (right) of the Test Problem 3.1 for k = 0.01.

The obtained numerical results from the mVIA-I for the Test Problem 3.2 are reported in Table 3. Results generated by mVIA-I show clear improvement over previous ones reported in [18], which indicate that the proposed algorithm is a powerful mathematical tool for getting an accurate numerical solution of the C-D-G equation. The results of the proposed algorithm (mVIA-I) are compared with the results of the modified Adomian decomposition method [18].
Thus, it is evident from the numerical results and comparison with the results of [18] that the mVIA-I is clearly defined and the results of the proposed algorithm are more accurate.

3.3. Test Problem 3

The Kaup-Kuperschmidt eq. (3), exploits the initial condition

$$w(x,0)=\frac{24k^2e^{kx}[4e^{kx}+e^{2kx}+16]}{[16e^{kx}+e^{2kx}+16]^2}, \tag{19}$$

and generates the exact solution given in [18] by

$$w(x,t)=\frac{24k^2e^{kx-t}\left[4e^{kx-k^5t}+e^{2kx-k^5t}+16\right]}{\left[16e^{kx-k^5t}+e^{2kx-2k^5t}+16\right]^2}. \tag{20}$$

The numerical results of Test Problem 3.3 generated using mVIA-I are illustrated graphically in Fig. 3.

The graphs included in Fig. 3 illustrate the comparison of approximate and exact solutions in terms of absolute errors of the Test Problem 3.3for k = 0.01 and different values of t.

Table 3. Comparison of numerical solutions of C-D-G equation (28) for the Test Problem 3.2.

x/t	t=0.4		t=0.8		t=5.0	
	mVIA-I	ADM [18]	mVIA-I	ADM [18]	mVIA-I	ADM [18]
2	1.01643×10^{-20}	3.11957×10^{-21}	6.77626×10^{-21}	1.78930×10^{-20}	3.38813×10^{-21}	2.48025×10^{-18}
4	6.77626×10^{-21}	2.70138×10^{-21}	1.01643×10^{-20}	2.01336×10^{-20}	1.35525×10^{-20}	5.70681×10^{-18}
6	6.77626×10^{-21}	5.50257×10^{-21}	3.38813×10^{-21}	3.91461×10^{-20}	6.77626×10^{-21}	8.93995×10^{-18}
8	3.38813×10^{-21}	4.74685×10^{-21}	0.00000	5.79897×10^{-20}	1.35525×10^{-20}	1.21797×10^{-17}
10	3.38813×10^{-21}	7.21049×10^{-21}	3.38813×10^{-21}	5.63358×10^{-20}	0.00000	1.54092×10^{-17}

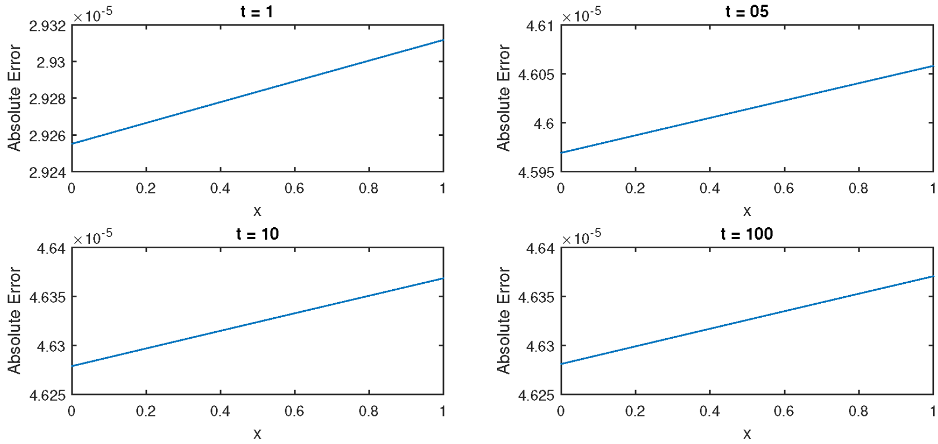

Fig. 3. Exact and approximate solution's comparison of the Test Problem 3.3 for different values of t and k = 0.01.

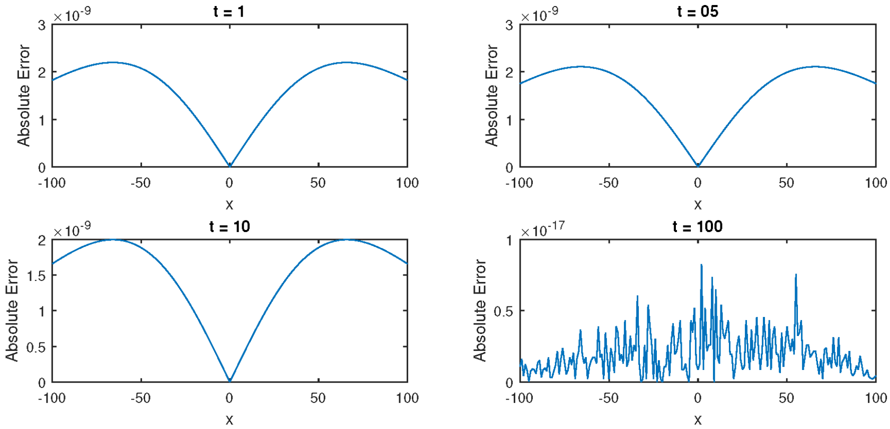

Fig. 4. Exact and approximate solution's comparison of the Test Problem 3.5 for t = 10 and k = 0.01 for the Test Problem 3.5.

3.4. Test Problem 4

The Lax's eq. (4) uses the initial condition

$$w(x,0) = 2k^2\left(2 - 3\tanh^2\left(k(x-c)\right)\right) \tag{21}$$

and possesses the exact solution given in [18] by

$$w(x,t) = 2k^2\left(2 - 3\tanh^2\left(k(x - 56k^4t - c)\right)\right), \tag{22}$$

where $\tanh(z)$ means the hyperbolic tangent function. From the numerical results of the Test Problem 3.4, generated using the mVIA-I and reported in Table 4 for different values of x and t, we conclude that the obtained results are pretty much agree with the exact solution and with the results given in [18]. The present method can be easily extended to fractal differential equations or fractional differential equations, where fractal variational theory [49],[50] can be used to identify the Lagrange multiplier.

3.5. Test Problem 5

The Ito eq. (5) starts from the following initial condition

$$w(x,0) = 20k^2 - 30k^2\coth^2(kx), \tag{23}$$

and has the exact solution given in [18] by

$$w(x,t) = 20k^2 - 30k^2\coth^2(kx - 96k^4 + x_0), \tag{24}$$

Table 4. Comparison of numerical solutions of Lax equation (4) for the Test Problem 3.4.

x/t	t=0.2		t=0.8		t=5.0	
	mVIA-I	[18]	mVIA-I	[18]	mVIA-I	[18]
2	5.74774×10^{-14}	5.76197×10^{-14}	2.29909×10^{-13}	2.30342×10^{-13}	1.43692×10^{-12}	1.42104×10^{-12}
4	1.14220×10^{-13}	1.15281×10^{-13}	4.56882×10^{-13}	4.60727×10^{-13}	2.85550×10^{-12}	2.84213×10^{-12}
6	1.69509×10^{-13}	1.72985×10^{-13}	6.78036×10^{-13}	6.19153×10^{-13}	4.23772×10^{-12}	4.26326×10^{-12}
8	2.22649×10^{-13}	2.30730×10^{-13}	8.90596×10^{-13}	9.21621×10^{-13}	5.56622×10^{-12}	5.68444×10^{-12}
10	2.72984×10^{-13}	2.88518×10^{-13}	1.09193×10^{-13}	1.15213×10^{-12}	6.82460×10^{-12}	7.10566×10^{-12}

Table 5. Comparison of numerical solutions of fKdV equation (6) for Test Problem 3.6.

x/t	E_2		E_4		E_6	
	mVIA-I	HPTM [35]	mVIA-I	HPTM [35]	mVIA-I	HPTM [35]
0.1	4.03000×10^{-07}	4.03001×10^{-07}	2.03168×10^{-10}	2.04000×10^{-10}	4.85726×10^{-14}	2.00000×10^{-12}
0.2	3.14615×10^{-06}	3.14615×10^{-06}	6.39581×10^{-09}	6.39800×10^{-09}	6.14129×10^{-12}	8.00000×10^{-12}
0.3	1.03656×10^{-05}	1.03656×10^{-05}	4.77886×10^{-08}	4.77890×10^{-08}	1.03657×10^{-10}	1.03000×10^{-10}
0.4	2.39942×10^{-05}	2.39942×10^{-05}	1.98186×10^{-07}	1.98186×10^{-07}	7.67223×10^{-10}	7.66000×10^{-10}
0.5	4.57809×10^{-05}	4.57809×10^{-05}	5.95330×10^{-07}	5.95333×10^{-07}	3.61487×10^{-09}	3.61500×10^{-09}

Table 6. Comparison of absolute errors of the mVIA-I and the Laplace decomposition method for sixth approximation for Test Problem 3.6.

t/x	x=1.0		x=1.5		x=2.5	
	mVIA-I	LDM [24]	mVIA-I	LDM [24]	mVIA-I	LDM [24]
0.01	0.00000	4.38198×10^{-11}	0.00000	3.32204×10^{-10}	3.55271×10^{-15}	4.25802×10^{-09}
0.02	8.88178×10^{-16}	7.22467×10^{-11}	8.88178×10^{-16}	5.47711×10^{-10}	3.55271×10^{-15}	7.02029×10^{-09}
0.03	1.19904×10^{-14}	1.19114×10^{-10}	1.95399×10^{-14}	9.03023×10^{-10}	4.97379×10^{-14}	1.15745×10^{-08}
0.04	8.79296×10^{-14}	1.96387×10^{-10}	1.43884×10^{-13}	1.48883×10^{-09}	3.94351×10^{-13}	1.90831×10^{-08}
0.05	4.19229×10^{-13}	3.23787×10^{-10}	6.90114×10^{-13}	2.45467×10^{-09}	1.87405×10^{-12}	3.14627×10^{-08}

Table 7. Comparison of numerical solutions of Kawahara equation (7) for the Test Problem 3.7.

x/t	mVIA-I	HPTM [35]
0.1	3.76856×10^{-09}	1.64944×10^{-06}
0.2	4.10160×10^{-08}	6.67437×10^{-06}
0.3	1.70439×10^{-07}	1.51864×10^{-05}
0.4	4.67929×10^{-07}	2.72977×10^{-05}
0.5	1.01400×10^{-06}	4.31169×10^{-05}

where $\coth(z)$ means the hyperbolic cotangent function. The numerical results generated using mVIA-I on the Test Problem 3.5 are reported graphically in Fig. 4. This figure shows the comparison of the approximate and the exact solutions of the Test Problem 3.5 for $t = 10$ and $k = 0.01$.

3.6. Test Problem 6

Consider the following fifth order KdV eq. (6)

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} - w\frac{\partial^3 w}{\partial x^3} + \frac{\partial^5 w}{\partial x^5} = 0, \tag{25}$$

which fulfils the initial condition

$$w(x,0) = e^x. \tag{26}$$

The exact solution of the fKdV eq. (25) with the condition (26) was given in [35] by

$$w(x,t) = e^{x-t}. \tag{27}$$

The numerical simulation for different values of times t and x of the Test Problem 3.6 using the mVIA-I are carried out in Table 5 and Table 6. Table 5 shows the comparison of the numerical results of the mVIA-I and the homotopic perturbation transform method (HPTM) from [35]. Numerical results in Table 5 are expressed in term of absolute errors for second (E_2), fourth (E_4) and sixth (E_6) order approximations. As the order of approximation increases, the order of accuracy increases and generated results converge to the exact solution. Therefore, numerical results show the efficiency and reliability of the mVIA-I method for different values of x and t. Table 6 comprises the comparison of the numerical results of the mVIA-I and the Laplace decomposition method [24].

3.7. Test Problem 7

Consider the Kawahara eq. (7)

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} - \frac{\partial^5 w}{\partial x^5} = 0, \tag{28}$$

satisfying the initial condition

$$w(x,0) = \frac{105}{169}\text{sech}^4\left(\frac{x-x_0}{2\sqrt{13}}\right), \tag{29}$$

where $\text{sech}(z)$ is the hyperbolic secant function. The exact solution to the eq. (28) with the condition (29) was given in [35] by:

$$w(x,t) = \frac{105}{169}\text{sech}^4\left(\frac{1}{2\sqrt{13}}\left(x - x_0 + \frac{36t}{169}\right)\right). \tag{30}$$

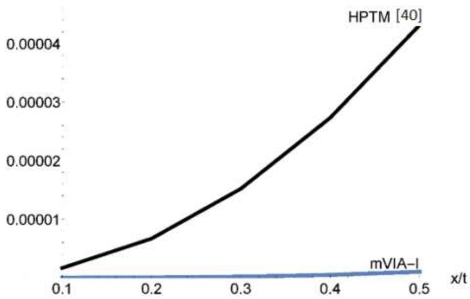

Fig. 5. Absolute error graphs for Test Problem 3.7.

To demonstrate the applicability and efficiency of our modified algorithm, we report the absolute errors for different values of x and t in Table 7. From the tabulated data, one can observe that the numerical and exact solutions by mVIA-I for the Kawahara eq. (7) for different time levels with $x = 6.0$, are in good agreement for the second-order approximation. Also, it is observable that the absolute errors generated by mVIA-I are much smaller than the corresponding absolute errors of the HPTM methods from [35]. Graphs included in Fig. 5 are evident confirmation of this observation.

4. Conclusion

This paper shows that the mVIA-I is very proficient, reliable and practically well suited for use in finding new traveling wave solutions for the higher-order differential equations. The reliability and accuracy of the method, as well as the decrease in the size of computational work, give this modified algorithm a more extensive pertinence. The results demonstrate that the algorithm is reliable, effective and gives more accurate solutions with respect to known methods. This modified algorithm facilitates computational work for solving nonlinear problems arising in applied sciences and engineering. High-accuracy solutions can be achieved in a few iterations of the proposed method compared to earlier techniques reported in the literature. We hope that the obtained results will be useful for further studies in scientific material science and engineering.

Author Contributions

T.A. Khan planned the scheme, initiated the project and suggested the problems; H. Ahmad analyzed the empirical results developed the mathematical modeling and examined the theory validation; P. Stanimirovic performed writing—review and editing; I. Ahmad performed formal analysis. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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References

[1] Ahmad, H., Khan, T. A., Durur, H., Ismail, G. M. & Yokus, A., Analytic approximate solutions of diffusion equations arising in oil pollution, *Journal of Ocean Engineering and Science*, 2020, doi: https://doi.org/10.1016/j.joes.2020.05.002.
[2] Ahmad, H., Khan, T. A. & Yao, S.-W., Numerical solution of second order Painlevé differential equation, *Journal of Mathematics and Computer Science*, 21, 2020, 150–157.
[3] GOLGELEYEN, F., HASDEMIR, M., NUMERICAL SOLUTION OF AN INVERSE PROBLEM FOR THE LIOUVILLE EQUATION, *TWMS Journal of Applied and Engineering Mathematics*, 9(4), 2019, 909-920.
[4] Yokus, A., Durur, H. & Ahmad, H., Hyperbolic type solutions for the couple Boiti-Leon-Pempinelli system, *Facta Universitatis, Series: Mathematics and Informatics*, 35, 2020, 523–531.
[5] Bazighifan, O., Ahmad, H. & Yao, S.-W., New Oscillation Criteria for Advanced Differential Equations of Fourth Order, *Mathematics,* 8, 2020, 728.
[6] El-Dib, Y., Stability Analysis of a Strongly Displacement Time-Delayed Duffing Oscillator Using Multiple Scales Homotopy Perturbation Method, *Journal of Applied and Computational Mechanics*, 4, 2018, 260–274.
[7] Ünsal, Ö. & Ma, W. X., Linear superposition principle of hyperbolic and trigonometric function solutions to generalized bilinear equations, *Computers and Mathematics with Applications*, 71, 2016, 1242–1247.
[8] Popov, V. L., Solution of adhesive contact problem on the basis of the known solution for non-adhesive one, *Facta Universitatis, Series: Mechanical Engineering*, 16, 2018, 93–98.
[9] Abouelregal AE, Ahmad H, Yao SW. Functionally Graded Piezoelectric Medium Exposed to a Movable Heat Flow Based on a Heat Equation with a Memory-Dependent Derivative. *Materials*.13(18), 2020, 3953.
[10] Ahmad I, Ahmad H, Thounthong P, Chu YM, Cesarano C. Solution of Multi-Term Time-Fractional PDE Models Arising in Mathematical Biology and Physics by Local Meshless Method, *Symmetry*, 12(7), 2020, 1195.
[11] Kawahara, T., Oscillatory solitary waves in dispersive media, *Journal of the Physical Society of Japan*, 33, 1972, 260–264.
[12] Gorshkov, K. A., Ostrovsky, L. A., Papko, V. V & Pikovsky, A. S., On the existence of stationary multisolitons, *Physics Letters A*, 74, 1979, 177–179.
[13] Yamamoto, Y. & Takizawa, É. I., On a solution on non-linear time-evolution equation of fifth order, *Journal of the Physical Society of Japan*, 50, 1981, 1421–1422.
[14] Djidjeli, K., Price, W. G., Twizell, E. H. & Wang, Y., Numerical methods for the solution of the third-and fifth-order dispersive Korteweg-de Vries equations, *Journal of Computational and Applied Mathematics*, 58, 1995, 307–336.
[15] Haupt, S. E. & Boyd, J. P., Modeling nonlinear resonance: A modification to the Stokes' perturbation expansion, *Wave Motion*, 10, 1988, 83–98.
[16] Darvishi, M. T. & Khani, F., Numerical and explicit solutions of the fifth-order Korteweg-de Vries equations, *Chaos, Solitons & Fractals*, 39, 2009, 2484–2490.
[17] Kaya, D., An application for the higher order modified KdV equation by decomposition method, *Communications in Nonlinear Science and Numerical Simulation*, 10, 2005, 693–702.
[18] Bakodah, H. O., Modified Adomain decomposition method for the generalized fifth order KdV equations, *American Journal of Computational Mathematics*, 3, 2013, 53.
[19] Wu, G. C. & He, J.-H., Fractional calculus of variations in fractal spacetime, *Nonlinear Science Letters A*, 1, 2010, 281–287.
[20] Khan, Y., An effective modification of the Laplace decomposition method for nonlinear equations, *International Journal of Nonlinear Sciences and Numerical Simulation*, 10, 2009, 1373–1376.
[21] Soltanalizadeh, B. & Branch, S., Application of differential transformation method for solving a fourth-order parabolic partial differential equations, *International Journal of Pure and Applied Mathematics*, 78, 2012, 299–308.
[22] Wazwaz, A.-M., Solitons and periodic solutions for the fifth-order KdV equation, *Applied Mathematics Letters*, 19, 2006, 1162–1167.
[23] Jafari, H. & Firoozjaee, M. A., Homotopy analysis method for solving KdV equations, *Surveys in Mathematics and its Applications*, 5, 2010, 89–98.
[24] Handibag, S. & Karande, B. D., Existence the solutions of some fifth-order kdv equation by laplace decomposition method, *American Journal of*

Computational Mathematics, 3, 2013, 80.
[25] Rafei, M. & Ganji, D. D., Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method, *International Journal of Nonlinear Sciences and Numerical Simulation*, 7, 2006, 321–328.
[26] Lei, Y., Fajiang, Z. & Yinghai, W., The homogeneous balance method, Lax pair, Hirota transformation and a general fifth-order KdV equation, *Chaos, Solitons & Fractals*, 13, 2002, 337–340.
[27] Yokus, A., Durur, H., Ahmad, H. & Yao, S.-W., Construction of Different Types Analytic Solutions for the Zhiber-Shabat Equation, *Mathematics*, 8, 2020, 908.
[28] Ahmad, H., Variational Iteration Method with an Auxiliary Parameter for Solving Telegraph Equations, *Journal of Nonlinear Analysis and Application*, 2018, 2018, 223–232.
[29] Ahmad, H., Khan, T. A. & Yao, S. W., An efficient approach for the numerical solution of fifth-order KdV equations, *Open Mathematics*, 18, 2020, 738–748.
[30] Ahmad, H., Auxiliary parameter in the variational iteration algorithm-II and its optimal determination, *Nonlinear Science Letters A*, 9, 2018, 62–72.
[31] Ahmad, H., Variational Iteration Algorithm-I with an Auxiliary Parameter for Solving Fokker-Planck Equation, *Earthline Journal of Mathematical Sciences*, 2, 2019, 29–37.
[32] Sawada, K. & Kotera, T., A method for finding N-soliton solutions of the KdV equation and KdV-like equation, *Progress of Theoretical Physics*, 51, 1974, 1355–1367.
[33] Caudrey, P. J., Dodd, R. K. & Gibbon, J. D., A new hierarchy of Korteweg--de Vries equations, *Proceedings of the Royal Society of London A: Mathematical and Physical Sciences*, 351, 1976, 407–422.
[34] Lax, P. D., Integrals of nonlinear equations of evolution and solitary waves, *Communications on Pure and Applied Mathematics*, 21, 1968, 467–490.
[35] Goswami, A., Singh, J. & Kumar, D., Numerical simulation of fifth order KdV equations occurring in magneto-acoustic waves, *Ain Shams Engineering Journal*, 9(4), 2018, 2265-2273.
[36] He, J.-H., A short review on analytical methods for a fully fourth-order nonlinear integral boundary value problem with fractal derivatives, *International Journal of Numerical Methods for Heat & Fluid Flow*, 2020, doi: 10.1108/HFF-01-2020-0060.
[37] Inokuti, M., Sekine, H. & Mura, T., *General use of the Lagrange multiplier in nonlinear mathematical physics*, Pergamon Press, Oxford, 1978.
[38] He, J.-H., Notes on the optimal variational iteration method, *Applied Mathematics Letters*, 25, 2012, 1579–1581.
[39] Ahmad, H., Seadawy, A. R. & Khan, T. A., Numerical solution of Korteweg--de Vries-Burgers equation by the modified variational iteration algorithm-II arising in shallow water waves, *Physica Scripta*, 95, 2020, 45210.
[40] Ahmad, H., Seadawy, A. R., Khan, T. A. & Thounthong, P., Analytic approximate solutions for some nonlinear Parabolic dynamical wave equations, *Journal of Taibah University for Science*, 14, 2020, 346–358.
[41] Ahmad, H. & Khan, T. A., Variational iteration algorithm-I with an auxiliary parameter for wave-like vibration equations, *Journal of Low Frequency Noise Vibration and Active Control*, 38, 2019, 1113–1124.
[42] Ahmad, H. & Khan, T. A., Variational iteration algorithm I with an auxiliary parameter for the solution of differential equations of motion for simple and damped mass–spring systems, *Noise & Vibration Worldwide*, 51, 2020, 12–20.
[43] Ahmad, H., Khan, T. A. & Cesarano, C., Numerical Solutions of Coupled Burgers' Equations, *Axioms*, 8, 2019, 119.
[44] Ahmad, H., Seadawy, A. R. & Khan, T. A., Study on numerical solution of dispersive water wave phenomena by using a reliable modification of variational iteration algorithm, *Mathematics and Computers in Simulation*, 2020, doi: https://doi.org/10.1016/j.matcom.2020.04.005.
[45] He, J.-H., A fractal variational theory for one-dimensional compressible flow in a microgravity space, *Fractals*, 2019, doi: 10.1142/S0218348X20500243.
[46] He, J.-H., Generalized variational principles for buckling analysis of circular cylinders, *Acta Mechanica*, 231, 2020, 899–906.
[47] He, J. H., Variational principle and periodic solution of the Kundu–Mukherjee–Naskar equation, *Results in Physics*, 17, 2020, 103031.
[48] Kaya, D. & El-Sayed, S. M., On a generalized fifth order KdV equations, *Physics Letters A*, 310, 2003, 44–51.
[49] He, C.-H., Shen, Y., Ji, F.-Y. & He, J.-H., Taylor series solution for fractal Bratu-type equation arising in electrospinning process, *Fractals*, 28(1), 2020, 2050011.
[50] He, J.-H., A simple approach to one-dimensional convection-diffusion equation and its fractional modification for E reaction arising in rotating disk electrodes, *Journal of Electroanalytical Chemistry*, 854, 2019, 113565.

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