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Research Paper

Different Groups of Variational Principles for Whitham-Broer-Kaup Equations in Shallow Water

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Abstract. Because the variational theory is the theoretical basis for many kinds of analytical or numerical methods, it is an essential but difficult task to seek explicit functional formulations whose extrema are sought by the nonlinear and complex models. By the semi-inverse method and designing trial-Lagrange functional skillfully, two different groups of variational principles are constructed for the Whitham-Broer-Kaup equations, which can model a lot of nonlinear shallow-water waves. Furthermore, by a combination of different variational formulations, new families of variational principles are established. The obtained variational principles provide conservation laws in an energy form and are proved correct by minimizing the functionals with the calculus of variations. All variational principles are firstly discovered, which can help to study the symmetries and find conserved quantities, and might find lots of applications in numerical simulation. The procedure reveals that the semi-inverse method is highly efficient and powerful, and can be extended to more other nonlinear equations.

Keywords: Variational principle, Whitham-Broer-Kaup equations, Nonlinear, Shallow water waves.

1. Introduction

Partial differential equations (PDEs) are usually used to describe nonlinear problems emerging in different scientific fields, such as physics, mechanics, biology, chemistry, meteorology, ocean and so on[1-3]. And numerous mathematical techniques have been developed to explore the approximate or exact solutions[4-23] for PDEs. Variational methods such as Ritz technique [14] and variational iteration method [15-19,23] have been, and continue to be, popular tools for nonlinear analysis. For example, the soliton solutions and their evolution of lots of nonlinear equations were accurately captured by the variational approximation method, which always substitutes some ansatzs into the obtained Lagrange functional, and find the the variational parameters by solving the corresponding Euler-Lagrange equations [14-19]. When contrasted with other analytical or numerical methods, variational-based methods show a lot of advantages[18-19]. Firstly, they can be used to investigate practical problems from a global perspective, and provide physical insight into the nature of the solutions. Secondly, they suggest an energy conservation for the whole solution domain, and require much less strong local differentiability of each variable than the methods, which solve PDEs directly. Last but not the least, the obtained solutions are the best among all the possible trial-functions. Because variational principles are the theoretical basis for many kinds of variational-based methods, it is very important to seek explicit variational formulations for nonlinear and complex PDEs, which is a nontrivial problem. Recently, many scientists have made many attempts and great success for constructing different kinds of variational principles in various fields such as fluid dynamics, meteorology, ocean, mathematical biology, solid state physics, and plasma physics [24-38]. In this paper, we will use the semi-inverse method to establish generalized variational principles for Whitham-Broer-Kaup (WBK) equations, which contain various solitary waves [38-44]. Although the WBK equations have been extensively studied for a long time by lots of scientists [38-44], but up to now variational principle for them has not been dealt with. The semi-inverse method was firstly proposed by the famous Chinese mathematician, Dr. Ji-Huan He [24-28]. It is broadly adopted to establish generalized variational principles directly from the governing equations, and the Lagrange crisis frequently encountered can be overcome because it is not necessary to introduce Lagrange multipliers [24-28]. The semi-inverse method has been used to search for variational formulations in plasma [25], mechanics [26], materials [27], fluid dynamics [28-32], solitary theory [34-37]. Recently, Wang et al. [34] established a generalized variational principle for a fractal PDE successfully, creating a new prospect on variational principles for fractal calculus [12,28,30,33-35] and fractional calculus [12,28,30,33-35] through the semi-inverse method.

2. Variation Principles for WBK Equations

In order to describe the propagation of shallow water waves, many well-known completely integrable models are introduced, such as KdV equation, Boussinesq equation, K-P equation, etc. Under Boussinesq approximation, Whitham, Broer and Kaup

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obtained the nonlinear coupled WBK equations [38-40]

$$\begin{aligned} &u_t + uu_x + H_x + \beta u_{xx} = 0 \\ &H_t + (uH)_x + \alpha u_{xxx} - \beta H_{xx} = 0 \end{aligned} \tag{1}$$

where $u = u(x,t)$ is the field of horizontal velocity, $H = H(x,t)$ is the height that deviates from the equilibrium position of liquid, and both of them are dimensionless variables. α and β are constants that represent different dispersive power. Correspondingly, the Eqs. (1) will be the model to describe distinct nonlinear wave. If $\alpha = 0$ and $\beta \neq 0$, WBK equations become classical long wave equations that describe shallow water wave with dispersions [41]; If $\alpha = 1$ and $\beta = 0$, WBK equations become the first type of variant Boussinesq equation [42]. If $\alpha = 0$ and $\beta = 1/2$, WBK equations become the approximate long wave equation that describe shallow water wave. Kaup[40] , Ablowitz [1] studied inverse transformation solution for the special case of Eqs. (l). Kaupershmidt [41] discussed their symmetries and conservation laws. By using improved homogenous balance method and Backlund transformation method, several families of exact analytical solution are derived for WBK equations, such as multiple solitary wave solutions, rational fraction solutions and periodic solutions, etc [42].

Our goals are to search for variational formulations whose Euler-Lagrange equations are equivalent to Eqs. (1). In order to use the semi-inverse method [24-30], we can rewrite Eqs. (1) in the following conservation form

$$\begin{aligned} &u_t + (u^2/2 + H + \beta u_x)_x = 0 \\ &H_t + (uH + \alpha u_{xx} - \beta H_x)_x = 0 \end{aligned} \tag{2}$$

According to the first equation of (2), we can introduce a special function Φ, defined as

$$\begin{aligned} &\Phi_x = u \\ &\Phi_t = -(u^2/2 + H + \beta u_x) \end{aligned} \tag{3}$$

so that the first equation of (2) is automatically satisfied. Similarly, in the light of the second equation of (2), another special function Π can be introduced as following:

$$\begin{aligned} &\Pi_x = H \\ &\Pi_t = -(uH + \alpha u_{xx} - \beta H_x) \end{aligned} \tag{4}$$

and the second equation of (2) is completely satisfied. Because finding Lagrangian formulations for Eqs. (1) is a nontrivial proble m, it is necessary to replace original variables by their representation as derivatives of potential fields. So, we try to construct different groups of variational principles (VPs) according to the above two set of equations (3) or (4), respectively.

2.1 The first group of VPs

In order to search for variational principles for the WBK equations, whose Euler-Lagrange equations satisfy Eqs. (2) and (3), we firstly give a trial-functional in the following form:

$$J(u,h,\Phi) = \iint L_1 dxdt \tag{5}$$

where L_1 is the trial-Lagrange functional. By the semi-inverse method [24-32], we assume that the trial-Lagrange functional can be written in the following different forms

$$L_1 = H\Phi_t + uH\Phi_x + \beta H\Phi_{xx} + \alpha u\Phi_{xxx} + F_1 \tag{6}$$

or

$$L_1 = H\Phi_t + (uH + \alpha u_{xx} - \beta H_x)\Phi_x + F_1 \tag{7}$$

and F_1 is an unknown function of u and H, and their derivatives. There are alternative approaches for construction of the trial-functional, see Refs. [24-32]. The advantage of the above trial-Lagrange functions of L_1 lies on the fact that the stationary condition with respect to Φ results in the following Euler-Lagrange equation:

$$\frac{\partial L_1}{\partial \Phi} - \frac{\partial}{\partial x}\left(\frac{\partial L_1}{\partial \Phi_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial \Phi_t}\right) + \frac{\partial}{\partial x^2}\left(\frac{\partial L_1}{\partial \Phi_{xx}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial L_1}{\partial \Phi_{xxx}}\right) = 0 \tag{8}$$

In view of (6) or (7), (8) is identical to the second one in WBK equations. Now calculating the stationary condition of the above trial-Lagrange functional L_1 with respect to u and H respectively, we obtain the following Euler equations:

$$\frac{\partial L_1}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L_1}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial u_t}\right) + \frac{\delta F_1}{\delta u} = 0 \tag{9}$$

$$\frac{\partial L_1}{\partial H} - \frac{\partial}{\partial x}\left(\frac{\partial L_1}{\partial H_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial H_t}\right) + \frac{\delta F_1}{\delta H} = 0 \tag{10}$$

where $\delta F_1 / \delta u$ and $\delta F_1 / \delta H$ is called the He's variational derivative [24-30] of F_1, defined as

$$\frac{\delta F_1}{\delta u} = \frac{\partial F_1}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F_1}{\partial u_x} - \frac{\partial}{\partial t}\frac{\partial F_1}{\partial u_t} + \frac{\partial^2}{\partial x^2}\frac{\partial F_1}{\partial u_{xx}} + \cdots$$

$$\frac{\delta F_1}{\delta H} = \frac{\partial F_1}{\partial H} - \frac{\partial}{\partial x}\frac{\partial F_1}{\partial H_x} - \frac{\partial}{\partial t}\frac{\partial F_1}{\partial H_t} + \frac{\partial^2}{\partial x^2}\frac{\partial F_1}{\partial H_{xx}} + \cdots$$

In view of Eq. (6) or (7), Eq.(9) and (10) can be re-written as follows:

$$H\Phi_x + \alpha\Phi_{xxx} + \frac{\delta F_1}{\delta u} = 0 \tag{11}$$

$$\Phi_t + u\Phi_x + \beta\Phi_{xx} + \frac{\delta F_1}{\delta H} = 0 \tag{12}$$

We search for such an F_1 so that Eqs. (11) and (12) turn out to be the field equations of (3). Therefore, substituting the equations (3) into (11) and (12) respectively, leads to

$$\frac{\delta F_1}{\delta u} = -uH - \alpha u_{xx} \tag{13}$$

$$\frac{\delta F_1}{\delta H} = H - u^2/2 \tag{14}$$

From the above equations (13) and (14), F_1 can be identified easily as follows

$$F_1 = \frac{H^2 - u^2 H + \alpha u_x^2}{2} \tag{15}$$

or

$$F_1 = \frac{H^2 - u^2 H - \alpha u u_{xx}}{2} \tag{16}$$

After substituting (15) or (16) into (6) and (7), respectively, four different trial-Lagrangian functions can be determined.

Finally, we construct the first group of four different variational principles successfully, for the WBK equations in shallow water, which read

$$J_1(u,H,\Phi) = \iint (H\Phi_t + uH\Phi_x + \beta H\Phi_{xx} + \alpha u\Phi_{xxx} + \frac{H^2 - u^2H + \alpha u_x^2}{2})dxdt \tag{17}$$

$$J_2(u,H,\Phi) = \iint [H\Phi_t + uH\Phi_x + (\alpha u_{xx} - \beta H_x)\Phi_x + \frac{H^2 - u^2H + \alpha u_x^2}{2}]dxdt \tag{18}$$

$$J_3(u,H,\Phi) = \iint (H\Phi_t + uH\Phi_x + \beta H\Phi_{xx} + \alpha u\Phi_{xxx} + \frac{H^2 - u^2H - \alpha u u_{xx}}{2})dxdt \tag{19}$$

$$J_4(u,H,\Phi) = \iint [H\Phi_t + uH\Phi_x + (\alpha u_{xx} - \beta H_x)\Phi_x + \frac{H^2 - u^2H - \alpha u u_{xx}}{2}]dxdt \tag{20}$$

Proof. Making any one of the above functionals, Eqs. (17)-(20), stationary with respect to three independent functions u, H and Φ severally, the following Euler-Lagrange equations can be obtained:

$$\delta\Phi: \quad -H_t - (uH)_x + \beta H_{xx} - \alpha u_{xxx} = 0 \tag{21}$$

$$\delta H: \quad \Phi_t + u\Phi_x + \beta\Phi_{xx} + H - u^2/2 = 0 \tag{22}$$

$$\delta u: \quad (H + \partial^2/\partial x^2)(\Phi_x - u) = 0 \tag{23}$$

in which $\delta\Phi$, δH and δu is the first-order variation for Φ, H and u. Obviously, the equation (21) is equivalent to the second equation in Eqs. (1). From the equation (23), we have $\Phi_x = u$, which is identical to the first equation in (3). Submitting $\Phi_x = u$ into Eq. (22) produces $\Phi_t = -u^2/2 - H - \beta u_x$, which is identical to the second equation in (3). So, we successfully proved the obtained four variational principles (17)-(20) correct.

2.2 The second group of VPs

In this part, we will search for another group of variational principles for the WBK equations by He's semi-inverse method [24-30]. In the same way, a trial-functional can be constructed as following:

$$J(u,H,\Pi) = \iint L_2 dxdt \tag{24}$$

where L_2 is the trial-Lagrange functional. On the basis of the semi-inverse method [24-32], we assume that the trial-Lagrange functional can be written as

$$L_2 = u\Pi_t + (H + u^2/2)\Pi_x - \beta u\Pi_{xx} + F_2 \tag{25}$$

or

$$L_2 = u\Pi_t + (H + u^2/2 + \beta u_x)\Pi_x + F_2 \tag{26}$$

and F_2 is an unknown function of u, H and their derivatives. The obvious merit of the above two trial-Lagrange functions lies on the fact that the stationary condition with respect to Π results in the following Euler-Lagrange equation:

$$\frac{\partial L_2}{\partial \Pi} - \frac{\partial}{\partial t}(\frac{\partial L_2}{\partial \Pi_t}) - \frac{\partial}{\partial x}(\frac{\partial L_2}{\partial \Pi_x}) + \frac{\partial^2}{\partial x^2}(\frac{\partial L_2}{\partial \Pi_{xx}}) = 0 \tag{27}$$

In view of (25) or (26), (27) becomes

$$-u_t - (H + u^2/2 + \beta u_x)_x = 0 \tag{28}$$

which is equivalent to the first equation in the system (1).

Now by calculating the stationary condition of the above trial-Lagrange functions L_2 with respect to u and H, respectively, we obtain:

$$\Pi_t + u\Pi_x - \beta\Pi_{xx} + \frac{\delta F_2}{\delta u} = 0 \tag{29}$$

$$\Pi_x + \frac{\delta F_2}{\delta H} = 0 \tag{30}$$

where $\delta F_2 / \delta u$ and $\delta F_2 / \delta H$ is called the He's variational derivative [24-30] of F_2, severally. In view of Eqs.(4), Eq.(29) and (30) can be re-written as follows

$$\frac{\delta F_2}{\delta u} = \alpha u_{xx} \tag{31}$$

$$\frac{\delta F_2}{\delta H} = -H \tag{32}$$

Because $\delta F_2 = \frac{\delta F_2}{\delta u}\delta u + \frac{\delta F_2}{\delta H}\delta H$, we can identify F_2 as follows

$$F_2 = -(H^2 + \alpha u_x^2)/2 \tag{33}$$

or

$$F_2 = -(H^2 - \alpha u u_{xx})/2 \tag{34}$$

Substituting (33) or (34) into (25) and (26), four new trial-Lagrange functions are generated.

At last, the second group of four variational principles are found for the WBK equations, which read

$$J_5(u,H,\Pi) = \iint [u\Pi_t + (H + u^2/2)\Pi_x - \beta u\Pi_{xx} - (H^2 + \alpha u_x^2)/2]dxdt \tag{35}$$

$$J_6(u,H,\Pi) = \iint [u\Pi_t + (H + u^2/2)\Pi_x + \beta u_x\Pi_x - (H^2 + \alpha u_x^2)/2]dxdt \tag{36}$$

$$J_7(u,H,\Pi) = \iint [u\Pi_t + (H + u^2/2)\Pi_x - \beta u\Pi_{xx} - (H^2 - \alpha u u_{xx})/2]dxdt \tag{37}$$

$$J_8(u,H,\Pi) = \iint [u\Pi_t + (H + u^2/2)\Pi_x + \beta u_x\Pi_x - (H^2 - \alpha u u_{xx})/2]dxdt \tag{38}$$

Proof. According to calculus rules of variations, we obtain the following Euler-Lagrange equations by making any one of the above four functionals, Eq. (35)-(38), stationary with respect to three independent functions Π, u and H.

$$\delta\Pi: \quad -u_t - (H + u^2/2)_x - \beta u_{xx} = 0 \tag{39}$$

$$\delta u: \quad \Pi_t + u\Pi_x - \beta\Pi_{xx} + \alpha u_{xx} = 0 \tag{40}$$

$$\delta H: \quad \Pi_x - H = 0 \tag{41}$$

in which $\delta\Pi$, δu and δH is the first-order variation of Π, u and H, respectively. Obviously, the equation (39) is equivalent to the first equation in Eq. (1). From the equation (41), we have $\Pi_x = H$, which is identical to the first equation in (4). Substituting $\Pi_x = H$ into Eq. (40) yields $\Pi_t = -H - u^2/2 - \beta u_x$, which is equivalent to the second equation in (4). Successfully, we prove the second group of four variational principles (35)-(38) correct. The above variational integral formulations provide conservation laws in an energy form for WBKs, and Can help us to understand deeply the physical relations and interactions between variables Π, u and H. And they also provide hints for numerical algorithms and possible solution structures for analytical methods for the discussed problem.

3. Discussions

In the second part, two different groups of variational principles (17)-(20), (35)-(38) were successfully constructed for the WBK equations by the semi-inverse method [24-32]. Moreover, the obtained variational principles were also proved correct by minimizing the corresponding functionals. From the results of research, it was concluded that the variational principle for the nonlinear equations studied in this paper is not unique, but has many different integral formulations. It is possible to construct new and distinct variational principles through choosing a different trial-Lagrange functional. The obtained variational principles can be further extended. By a combination of different formulations of (17)-(20), (35)-(38), new families of variational principles can be established as follows:

$$J(u,H,\Phi)=\iint[H\Phi_t+uH\Phi_x+w_1(\alpha u\Phi_{xxx}+\beta H\Phi_{xx})+(1-w_1)(\alpha u_{xx}-\beta H_x)\Phi_x+\frac{H^2-u^2H+\alpha u_x^2}{2}]dxdt \tag{42}$$

and

$$J(u,H,\Phi)=\iint[H\Phi_t+uH\Phi_x+w_2(\alpha u\Phi_{xxx}+\beta H\Phi_{xx})+(1-w_2)(\alpha u_{xx}-\beta H_x)\Phi_x+\frac{H^2-u^2H-\alpha uu_{xx}}{2}]dxdt \tag{43}$$

and

$$J(u,H,\Pi)=\iint[u\Pi_t+(H+u^2/2)\Pi_x-w_3\beta u\Pi_{xx}+(1-w_3)\beta u_x\Pi_x-\frac{H^2+\alpha u_x^2}{2}]dxdt \tag{44}$$

and

$$J(u,H,\Pi)=\iint[u\Pi_t+(H+u^2/2)\Pi_x-w_4\beta u\Pi_{xx}+(1-w_4)\beta u_x\Pi_x-\frac{H^2-\alpha uu_{xx}}{2}]dxdt \tag{45}$$

where $w_i\in[0,1]$, $i=1,2,3,4$, are weight factors. From the stationary conditions of Eqs. (42)-(43), we can obtain the Euler-Lagrange equations, which are equivalent to Eqs. (1) and (3). The process of correctness proving is exactly same as the first groups of VPs. By taking the variations of variational formulations (44)-(45), we can get the Euler-Lagrange equations, which are identical to Eqs. (1) and (4). The detail process of proving is exactly same as the second groups of VPs. So, it is obvious that Eqs. (42)-(45) are also the variational principles of WBK equations. If we set $w_i=1$, $i=1,2,3,4$, variational principles (42)-(45) become (17), (19), (35) and (37) correspondingly. If we set $w_i=0$, $i=1,2,3,4$, variational principles (42)-(45) are transformed into (18), (20), (36) and (38) in turn. So, we can conclude that all the generalized variational principles appeared in the second part are only special cases of the compound functionals (42)-(45) by appropriately choosing different weight parameters.

4. Conclusions

From the above analysis, it was concluded that the variational principle for WBK equations is not unique, but has many integral formulations. Based on the above variational principles, on the one hand, we can study possible solution structures for solitary waves of the WBK equations and understand the physical relations and interactions between horizontal velocity and wave height deeply. On the other hand, they also provide hints for numerical algorithms, then the WBK equations (1) can be solved numerically by variational methods. In the analytical analysis and numerical simulations, it is of great importance to choose an appropriate variational principle from (42)-(45) according to practical needs. In this paper, it was also revealed that the semi-inverse method is straightforward and powerful, which can be widely applied to other nonlinear equations arising in mathematical physics and nonlinear sciences. It should be noted that the research frontier of the semi-inverse method becomes how to construct variational principle for fractal problems so far. For example, He [28, 30, 33] and Wang et al. [34] and successfully obtained a variational principle for nonlinear problems in the fractal space. Similarly, our work can be extended to fractal calculus in the future, in which WBK equations with unsmooth boundaries will be considered.

Author Contributions

All authors have important contributions in this paper. The details are as following. Conceptualization, X.Q. Cao; Methodology, X.Q. Cao and Y.N. Guo; Writing – original draft, X.Q. Cao; Writing – review & editing, Y.N. Guo, S.C. Hou and C.Z. Zhang. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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