Journal of

Applied and Computational Mechanics

Shahid Chamran
University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

Reduction in Space for Dynamic Finite Element Analysis of Assemblies of Beam-columns when the Mass is Available in Digitized Format

Aram Soroushian[1], Saeed Amiri[2]

[1] Structural Engineering Research Centre, International Institute of Earthquake Engineering and Seismology,
No. 21, West Arghavan, North Dibajee, Farmanieh, Tehran 19537, Iran, Email: a.soroushian@iiees.ac.ir, aram.soroushian@gmail.com
[2] Structural Engineering Research Centre, International Institute of Earthquake Engineering and Seismology,
No. 21, West Arghavan, North Dibajee, Farmanieh, Tehran 19537, Iran, Email: saeed.amiri.saeed@gmail.com

Abstract. In 2008, a technique was proposed to reduce run-times in analysis of semi-discretized equation of motion against dynamic excitations available in digitized format. Later, the technique was successfully adapted to reduce numbers of degrees of freedom in finite element analysis of assemblies of beam-columns subjected to static digitized loads. In this paper, attention is paid to dynamic finite element analysis of assemblies of beam-columns. It is shown that, when the mass is available in digitized format, after small modifications in the original technique, the adaptation can simplify the analysis, regardless of the models' sizes, their linearity or non-linearity, and whether the damping is classical or non-classical. The reductions in run-time and in-core memory are considerable, while the changes in accuracy can be negligible.

Keywords: Reduction in space, Dynamic finite element analysis, Beam-columns, Accuracy, Computational effort.

1. Introduction

By invention of computers in the mid-twentieth century, approximate numerical computations became popular in different areas of science and engineering. The major necessity of numerical computations, leading to the results desired accuracy, is the errors convergence to zero [1, 2]. Numerical stability and computational effort are other important concerns, e.g. see [3, 4]; and in practical computations, the main interests are sufficient accuracy and less computational effort, e.g. see [5].

Step-by-step solution of ordinary initial value problems is a broadly accepted computation that covers many approximate computational methods [6, 7]. For these methods, an important application area is analysis of initial value problems implying structures' semi-discretized equations of motion [3, 8, 9], typically stated as:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{f}_{int} = \mathbf{f}(t) \qquad 0 \leq t \leq t_{end}$$

$$
\begin{aligned}
\mathbf{u}(t=0) &= \mathbf{u}_0 \\
\dot{\mathbf{u}}(t=0) &= \dot{\mathbf{u}}_0 \\
\mathbf{f}_{int}(t=0) &= \mathbf{f}_{int_0} \\
\mathbf{Q} &\leq \bar{\mathbf{O}}
\end{aligned}
\tag{1}
$$

In eq. (1), $\mathbf{M}$ is the mass matrix, $\mathbf{u}$ is the vector of unknown displacements, each top dot represents once differentiation with respect to time, $\mathbf{f}_{int}$ is the vector of unknown internal forces, $\mathbf{f}(t)$ is the vector of external forces, $\mathbf{Q} \leq \bar{\mathbf{O}}$ indicates restrictions because of nonlinearity (e.g. the inequalities defining simple impacts between masses of multi-body systems, or poundings between adjacent buildings [10]), t and t_{end} stand for time and the analysis duration respectively, and "0" as a right subscript denotes that the argument is at its initial status. Step-by-step solution of eq. (1), addressed as direct time integration or time history analysis in structural dynamics [3, 9], is pictorially summarized in Fig. 1. In analysis of eq. (1) by direct time integration, when $\mathbf{f}(t)$ has a digitized format, an appropriate integration step depends on the step in which $\mathbf{f}(t)$ is digitized; see [3, 11].
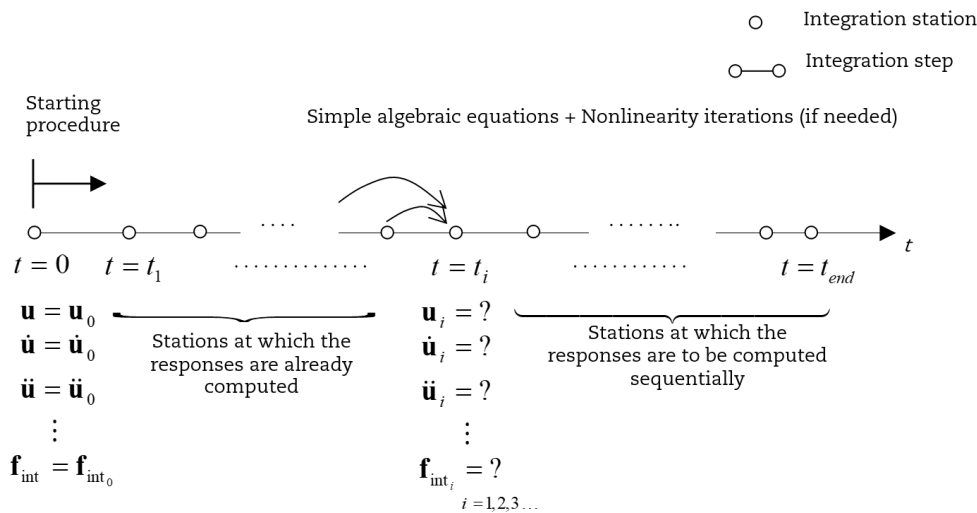
Fig. 1. A pictorial description of step-by-step analysis of eq. (1) [3]

Besides, based on Taylor series expansion [12], for a function F smooth [12, 13] with respect to a continuously changing independent variable ξ,

$$F\left(\xi \pm \delta\right) = F\left(\xi\right) \pm \delta \frac{\partial F\left(\xi\right)}{\partial \xi} + \frac{1}{2!}\delta^2 \frac{\partial^2 F\left(\xi\right)}{\partial \xi^2} + \cdots \tag{2}$$

where, ξ is the independent variable and δ stands for an arbitrary small change in ξ. Accordingly,

$$F(\xi + \delta) + F(\xi - \delta) = 2F(\xi) + O\left(\delta^2\right), \tag{3}$$

and the expression

$$aF\left(\xi\right) + \left(1 - a\right)\left\{b_1\left[F\left(\xi + \delta_1\right) + F\left(\xi - \delta_1\right)\right] + b_2\left[F\left(\xi + \delta_2\right) + F\left(\xi - \delta_2\right)\right] + \cdots + b_{n_b}\left[F\left(\xi + \delta_{n_b}\right) + F\left(\xi - \delta_{n_b}\right)\right]\right\}, \tag{4}$$

converges to $F(\xi)$ with second order of accuracy, when $\delta_{i=1,2,\ldots,n_b}$ converges to zero. In eq. (4), $\delta_{i=1,2,\ldots,n_b}$ represents arbitrary small changes in ξ, and a and $b_{i=1,2,\ldots,n_b}$ satisfy

$$|a| < \infty \quad , \quad \left|b_{i=1,2,\ldots n_b}\right| < \infty \quad , \quad \sum_{i=1}^{n_b} b_i = 0.5 \tag{5}$$

Meanwhile, in step-by-step solution of eq. (1), if the integration scheme is single-solution [4, 14], and the excitation $\mathbf{f}$ is inexact, the second order convergence of $\mathbf{f}$ to its exact value will not impair the second order convergence of the response [15, 16]. Finally, natural phenomena, e.g. ground motions, are generally smooth with respect to time, and available in digitized format; see [17]. Considering these, in 2008, a technique was proposed, for faster analysis of semi-discretized equations of motion against digitized excitations [15]. The technique replaces the excitation $\mathbf{f}$ with an excitation digitized in larger step, such that separate analyses against the two excitations, using the two digitization steps as the integration steps, lead to close responses. The effort essential for the replacement is negligible. Accordingly, implementation of the technique results in less computational effort in the price of negligible change in the response accuracy. Later, the technique, which was originally proposed for enlargement of digitization step by integer scales [15], was extended to enlargements by real scales [18]; formulations of both techniques are summarized in the appendix.

For adequacy of the technique's performance from the standpoint of accuracy [3, 15, 18], the digitization step should be smaller than needed for the response accuracy. In view of the existing literature (e.g. see [3, 11]), this implies:

$$_f\Delta\tilde{t} \leq \text{Min}\left(\frac{T}{\chi}, \Delta t_{cr}, \Delta t_r\right) \tag{6}$$

In eq. (6), $_f\Delta\tilde{t}$ is the digitization step of the new excitation (the digitization step of the original excitation is denoted by $_f\Delta t$), T is the smallest worthwhile period of the response [3, 19] or an approximation of this period [11], Δt_{cr} implies the largest integration step providing numerically stable responses [4], Δt_r stands for the largest step we accept as the response digitization step [3], and specific for each analysis (linear/nonlinear without impact/nonlinear with impact) [3, 11],

$$\chi = \begin{cases} 10 & \text{when the behavior is linear} \\ 100 & \text{when the behavior is nonlinear but not involved in impact} \\ 1000 & \text{when the nonlinear behavior is involved in impact} \end{cases} \tag{7}$$

Though ambiguities exist in the notion and availability of T and χ prior to the analysis, the technique's performance in the past experiences was successful; see Table 1.

Table 1. Samples of tests carried out to study the technique [15, 18] when implemented in transient analysis of semi-discretized systems against digitized excitations

System	Effort reduced, without worth-while reduction in accuracy (%)	Details
A thirty storey building	50	Two excitations are taken into account
Bridges	30-80	About 20 bridges with linear and nonlinear behaviors, some with pre-stressed elements and some subjected to multi-support excitation, against different excitations
Power station, Cooling tower, Space frame, Silo	>50	One or two of each special structure, considering linear and nonlinear behavior and different near-field and far-filed excitations and different integration schemes
Milad tele-communication tower	50-70	Considering linear and nonlinear behavior and different near-field and far-field excitations and different integration schemes

Table 2. Tests carried out to study the technique [15, 18] when implemented in finite element analysis of beam-columns assemblies against static loadings available in digitized format

Example	Effort reduced without worthwhile reduction in accuracy (%)
A cantilever beam	96
Beam-columns with loading non-perpendicular to the member axis in linear / nonlinear analysis	95 / 96
Curved beam-column in linear / nonlinear analysis	98 / 98
Realistic example (load changing with low / high frequency)	98 / 82

Reductions in computational effort because of the technique can be considerable (see Table 1 and [3]). Implementation of the technique is computationally cheap and easy, and the technique can be plugged in the pre-processing stage of arbitrary analysis software [3]. Additionally, as observed twice, the negligible change of accuracy does not necessarily imply loss of accuracy [3]. Considering these, extension/adaptation of the technique's application to other areas started in 2017 [20]. In static analysis of finite element models of assemblies of beam-columns, when a part of the static load is distributed and available in digitized format, the technique [15, 18] can be simply implemented. The result is reduction of the analysis run-time (the case in the original application [15, 18]), as well as the in-core memory [20, 21]. In more detail, in the first application of the technique [15, 18], i.e. transient analysis of semi-discretized models against digitized excitations, reduction in computational effort is because of enlargement of integration step (reduction in time) and the run-time is being affected. In the second application, the reduction in computational effort is because of reduction in the number of degrees of freedom (reduction in space), and both the run-time and the in-core memory are being affected. The expression "reduction in space" is used above and in the title of this paper to indicate the distinction between enhancement in analysis efficiency due to enlargement of digitization step along a spatial axis (and the resulting decrease in the number of degrees of freedom) and enhancement in analysis efficiency due to enlargement of digitization step along the time axis (and the resulting larger time integration step). In this paper, while the former is addressed as "reduction in space", the latter, not under consideration here, is addressed as "reduction in time". (For an arbitrary dynamic finite element model, "reduction in space" implies reduction of the structure's number of degrees of freedom, while no simplification is applied along the time axis). There are two more differences between the two applications.

First, while the integration interval may be slightly elongated in the original application (transient analysis of semi-discretized models) (see the appendix), the records lengths and the geometry need to be kept unchanged in implementation in static analysis of beam-columns' assemblies [21]. This difference can be simply eliminated by using slightly smaller beam-columns elements. The second difference is that in implementation of the technique [20, 21] in static analysis of beam-column assemblies' the digitized records shall express the load intensity, not the load values. This is while, in the original application, the records express the excitation values [15] (the reason mainly originates in considerations of the static load record in finite elements and the excitation record in time integration). This difference can also be simply eliminated, prior to the technique's implementation [21]. Considering these, several examples were studied for the second application [20, 21]; see Table 2.

Regarding efficient finite element analysis, several approaches are addressed in the literature; model reduction, adaptive mesh generation, and parallel processing, are among the notable approaches; see [5, 22-25]. Concentrating on model reduction in dynamic finite element analysis, dynamic condensation, sub-structuring, and component mode synthesis, are of the popular techniques [26-28], in close relation with practical applications, such as structural dynamic modification [29, 30]. The achievements exemplified in Table 2 can be considered as examples of a conceptually new approach for model reduction (with no direct dependence to stiffness, damping, and mass matrices), which though is now limited to analysis of assemblies of beam-columns (when the load is static and available in digitized format), the resulting reduction in computational effort is considerable (see [3, 5, 20-32]), and the change in accuracy is negligible. These plus ease and cheapness of implementation, simple applicability in different structural analysis software, almost identical formulation in the two applications [3, 15, 20, 21], and conceptually similar performance in linear and nonlinear problems (see Table 2), explain further study on the achievements reported in [3, 15, 18, 20, 21]. Specially note that different from many model reduction methods with application in finite elements [5], the positive points addressed above, appear together in the second application of the technique [15, 18, 20, 21]. Considering these, the objective in this paper is to extend/adapt the recent application, i.e. static finite element analysis of assemblies of beam-columns [20, 21], to a third application, i.e. dynamic finite element analysis of a special class of beam-columns assemblies. In more detail, the objective is to study the application of the technique first proposed for accelerating time history analysis [15, 18], in dynamic finite element analysis of assemblies of beam-columns, when the mass is in a digitized format. The main attributes of the study are:

(1) The behavior to be studied is the dynamic behavior,
(2) The structures under consideration are assemblies of beam-columns,
(3) The records (digitized in constant step) to be converted are the beam-columns' digitized masses.

Seismic analysis of underground lengthy structures, e.g. tunnels, is a practical application; see [21, 33]. After this introduction, the adaptation of the technique [15, 18] is explained, including the key idea, the computational procedure, and some discussions on the good accuracy. Three examples are presented later, and after brief discussion on the practical significance and future perspective, the paper is concluded with a set of the most important achievements.

2. Theory

2.1 Key idea

Consider a semi-discretized model of a beam-columns' assembly, such that the mass is totally or partially available in digitized format and a dynamic load is applied to the assembly. For dynamic finite element analysis of such models, the sizes of beam-column elements are generally being set with attention to the mass digitization step. Accordingly, the number of the elements and the analysis computational effort can be excessive. A key idea to reduce this computational effort by using the technique proposed in [15, 18] is stated in this section.

The key idea is very simple. It is using the formulation presented in the appendix [15, 18] (with few modifications), to replace the digitized mass with a mass digitized in larger steps, and applying ordinary dynamic finite element analysis. This implies no specific new formulation for adaptation of the technique proposed in [15, 18] to dynamic finite element analysis of the assemblies of beam-columns. A similar situation was observed when adapting the technique [15, 18] to the static finite element analysis [20, 21]. (The almost identical formulation can indeed be considered as an advantage of the original technique [15, 18] entailing its easy adaptation to different applications.)

To better explain the key idea, consider the details of the original technique [15, 18] and the explanations presented for its adaptation to static finite element analysis of beam-column assemblies subjected to static digitized loads [20, 21] (also see the appendix and Section 1). Specially note that the backbone of the step-enlargement technique [3, 15, 18] is responses convergence to the exact responses, which is a necessity for typical approximate computation [1, 3, 34]. Besides, take into account that, in both transient analysis of semi-discretized equations of motion, and static finite element analysis of the beam-column assemblies, the technique can be implemented prior to the analysis. Despite the similarities, in static finite element analysis of the beam-column assemblies, some additional considerations were needed, to address differences between step-by-step analysis of initial value problems and finite element analysis of boundary value problems [4, 6, 14]. The resulting adaptation [21] was successful (see Table 2). Accordingly, in dynamic finite element analysis of beam-columns' assemblies when the masses are available in digitized format, an approach close to that stated in [20, 21] is plausible. Based on the convergence necessity in finite element analysis, the good accuracy can be anticipated, as well. The main modification originates in that whether the mass model is lumped or distributed. Modifications are to be set in order to directly implement the adaptation proposed in [20, 21] in analysis of dynamic finite elements models of the beam-columns' assemblies. The modifications and a computational procedure are presented next, explanations on sufficiency of the response accuracy are stated later, and the performance is studied in Section 3.

2.2 Formulation modification and computational procedure

The main formulation of the adaptation proposed in this paper is identical to that of the original technique [15, 18]; see the appendix. The small differences are explained in continuation. When the mass is lumped, the amount of the mass between each two sequential lumped masses addressed by the digitized record is zero (independent of the type of the finite elements used). This is different from the past applications [15, 18, 20, 21], where the dynamic excitation between the excitation stations is not zero, or the static load is continuous along the beam-columns longitudinal axes. Therefore, when the lumped mass record is being replaced (according to the formulation presented in the appendix) with a record digitized in larger steps, the total mass decreases drastically. The resulting response of the structural analysis can be completely erroneous. To overcome this problem, the formulations and discussions presented in the appendix and [15] are reviewed in detail. The consequence is the essentiality to multiply the record produced by the technique [15, 18] by the enlargement ratio below:

$$n = {}_f \frac{\Delta \tilde{x}}{{}_f \Delta x} \tag{8}$$

(${}_f \Delta x$ and ${}_f \Delta \tilde{x}$ correspond to ${}_f \Delta t$ and ${}_f \Delta \tilde{t}$, however in the space domain, i.e. while ${}_f \Delta t$ and ${}_f \Delta \tilde{t}$ are used in the application of the technique to transient analysis, ${}_f \Delta x$ and ${}_f \Delta \tilde{x}$ are the corresponding variables in application of the technique to finite element discretization; see also [20, 21]). The above modification can be explained with attention to the considerations taken into account in the formulation presented in the appendix. Specifically, the formulation is set considering that in the structural analysis the $\mathbf{f}$ can be nonzero (even though unknown) at temporal/spatial locations different from the digitization stations. This is not the case for lumped mass models. Accordingly, to use the formulation presented in the appendix for an $\mathbf{f}$ representing lumped mass, it is reasonable to consider each data of the record distributed uniformly throughout ${}_f \Delta x$. This implies replacement of $\mathbf{f} / {}_f \Delta x$ instead of $\mathbf{f}$ in the formulation in the appendix. Similarly, the consequence of the formulation presented in the appendix, $\tilde{\mathbf{f}}$, though is computed for specific spatial locations, is not necessarily zero elsewhere; see the past two applications [15, 18, 20, 21]. The case is different for dynamic finite element analysis considering lumped masses. Accordingly, it is reasonable to replace the $\tilde{\mathbf{f}}$ with $\tilde{\mathbf{f}}/{}_f \Delta \tilde{x}$, as well. In view of Eq. (8) and the linearity of the formulations presented in the appendix, the above-mentioned two replacements imply n times larger $\tilde{\mathbf{f}}$, when $\mathbf{f}$ stands for lumped mass, i.e.

$$\tilde{\mathbf{M}} = n \tilde{\tilde{\mathbf{M}}} \qquad \text{(when the mass is lumped)} \tag{9}$$

In Eq. (9), $\tilde{\mathbf{M}}$ stands for the lumped mass to be used in the new (reduced) analysis, and $\tilde{\tilde{\mathbf{M}}}$ is the lumped mass (in digitized format) computed by applying the formulation presented in the appendix to the original mass $\mathbf{M}$. Additional error is induced to the computation by the above-mentioned division by the digitization step. However, the approximation is reasonable and at least of the order of the approximation induced by the technique [15, 18] and hence is negligible.

When the mass is originally distributed and the record $\mathbf{f}$ represents the intensity of the mass (also in consistence with the study reported on static analysis of beam-columns assemblies [20, 21]):

$$\tilde{\mathbf{M}} = \tilde{\tilde{\mathbf{M}}} \qquad \text{(when the mass is originally distributed and expressed as mass intensity)} \tag{10}$$

The case of distributed mass expressed as values of mass at digitization stations is very rare. Finally, it is worth noting that, in structural analysis practice, lumped mass models are more popular than distributed mass models, e.g. see [35].

For implementation of the technique [15, 18] in dynamic finite element analysis, another point, originating in the type of the problem, i.e. boundary value problem versus initial value problem, should be taken into account as well. It is to prevent changes in the beam-columns' lengths. In other words, the case $x'_{end} \neq x_{end}$, addressed in the appendix, would rather be prohibited, in order to eliminate additional nonlinearities due to changes in geometry. This can be materialized, by using slightly smaller

elements, throughout beam-columns undergone change of length because of implementation of the technique.

In view of the above explanations, a computational procedure to simplify analysis of the dynamic finite element models, is stated below:

1. Using the existing experience or a preliminary simple analysis, select the integration step and beam-column element sizes, such that to accurately define the response, taking into account the digitization step of the mass record (i.e. not using elements larger than the mass digitization step for beam-columns with digitized mass). Note that analysis using these selections implies ordinary analysis.
2. Repeat Step 1, while disregarding the restriction because of the mass digitization steps.
3. For each beam-column, divide the element length obtained in Step 2 to that obtained in Step 1.
4. Assign a positive value larger than one to n, taking into account the results computed in Step 3. (When the results are smaller than one, the reduction may be unsuccessful; generally, this is not the case.)
5. Using the value assigned to n, enlarge the beam-column elements computed in Step 1, while replacing the mass, available in digitized format, with masses digitized in n times larger steps, by
(a) Using the formulation in the appendix and the value assigned to n in Step 4,
(b) Implementation of either eq. (9) or eq. (10)
6. The lengths of the masses records obtained in Step 5 are either unchanged or slightly lengthier than the original records. In the latter case, the size of the elements on the beam-columns with mass should be decreased (by multiplying them in the record's x_{end}/x'_{end} (for x'_{end} see the appendix). (The amount of the decrease is small and need to be separately computed and applied for each beam-column with digitized mass.)
7. Because of the enlargement addressed in Step 5, the lengths of the without-mass beam-columns may change. In such a case, for each beam-column, the lengths of the elements should decrease uniformly and the least, causing the change in the beam-column's length to disappear.

Regarding Steps 1 and 2, it is worth noting that efforts to simplify these two steps are ongoing; e.g. see [36]. Meanwhile, for nonlinear analyses, selection of nonlinearity parameters, e.g. nonlinearity tolerance, is essential, in Step 1.

2.3 On the good performance

Performance of a method proposed for enhancing the efficiency of structural analysis can be considered adequate, if it is applicable to a broad range of problems, its implementation is easy and cheap, the reduction in computational effort is considerable, and the loss of accuracy is negligible. In view of Section 2.2 and the appendix, the ease and low cost of implementation are obvious for the adaptation proposed in this paper. With regard to versatility, computational effort and responses' accuracy, the mainstay of the study is the examples presented in Section 3. Still, brief theoretical explanations are presented in continuation.

For computational effort, when the analysis is linear, reduction in the computation effort is clear. By implementation of the proposed adaptation, sizes of the structural matrices/vectors reduce, without any change in the time integration step. This entails reduction in run-time and in-core memory (reduction in in-core memory is more). For nonlinear analyses, reduction in computational effort cannot be guaranteed, due to the probable effects of nonlinearity iterations [37]. However, because of the generally more influence of matrices/vectors sizes on the computational effort (compared to the nonlinearity iterations), the reduction is also likely when the analysis is nonlinear.

For accuracy, consider dynamic finite element analysis of a beam-column assembly and the relation below (see [38]):

$$U^a = F\left(P_{i=1,2,...n_p}, \lambda_1, \lambda_2\right) \tag{11}$$

between an arbitrary response obtained from the analysis U^a, real physical variables contributing the analysis $P_{i=1,2,...n_p}$, λ_1 as an indicator for sizes of the finite elements throughout the structural system, λ_2 as an indicator for sizes of the time steps throughout the integration interval, and F as a schematic representation of the computation. For the experiences reported in Table 1, the technique [15, 18] causes changes in λ_2 without changing λ_1. In this study, the technique causes changes in λ_1 without changing λ_2. In ordinary analysis, $P_{i=1,2,...n_p}$ undergoes no change and changes of λ_1 or λ_2 (not because of the technique [15, 18]) generally cause second order convergence in U^a. That is implementation of either finite elements or time integration leads to second order accuracy for the responses of linear analyses [4, 14, 39, 40]. Accordingly, in ordinary linear dynamic finite element analyses, eq. (11) implies similar roles for λ_1 and λ_2, from the point of view of convergence. Implementation of the technique [15, 18], either in its original application [3, 15, 18] or analysis of dynamic finite element models of beam-columns assemblies, causes errors in U^a, additional to ordinary analysis. These errors are associated with the changes of λ_2 or λ_1 and the changes included in $P_{i=1,2,...n_p}$. Since, based on the convergence back-bone of the technique [15, 16], these additional errors converge with the second order of accuracy, implementation of the technique and the above-mentioned additional errors cannot change the similarity of the roles of λ_1 and λ_2 in eq. (11). Therefore, with attention to the fact that in approximate computations accuracy is being studied under the umbrella of convergence [1, 34], and the original application of the technique proposed in [13, 18] leads to responses with sufficient accuracy (see Table 1), it is reasonable to expect good accuracy for the responses in the second application. That is when $P_{i=1,2,...n_p}$ and λ_1 change, according to the details of the technique [15, 18], and the modifications addressed in Sections 2.1 and 2.2 (and λ_2 does not change), the computed responses can be sufficiently accurate. For nonlinear analyses, the second order convergence need to be preserved; see [41].

3. Illustrative examples

3.1 Introduction

To show the good performance of the proposed adaptation, the study of three examples is reviewed in this section. Since, the examples were the first to study reduction in space of dynamic finite element analysis by the technique proposed in [15, 18], there was no concern on the largest value acceptable for n. It sufficed to show that computational effort can be reduced without worthwhile change in the response accuracy. The application area was structural analysis against ground motions applied uniformly to all supports of the structure [42]. The OpenSees (Open System for Earthquake Engineering Simulation) software [35] was used for the analysis, and the S.I. system was set for the units.

3.2 Simple example consisting of a beam a column and a beam-column

Consider the structural system introduced in Fig. 2(a) and Table 3. The only mass is the lumped mass of Member BC; see Fig. 2(b). The dynamic excitation is addressed in Fig. 2(c). Linear two-node beam-column elements were used for discretization in space [35, 39, 40], and the average acceleration method [43] (with steps equal to the excitation steps, i.e. 0.02 seconds) was used for time integration. The analysis was first carried out, using elements with lengths equal to 0.01 m, $0.01\sqrt{2}$ m, and 0.01 m, for members AB, BC, and CD respectively, and then repeated, using eight times larger elements, according to the procedure introduced in Section 2.2; for the responses, see Fig. 3. The run-time decreased from 272 seconds to 17.1 seconds, and the number of degrees of freedom decreased from 5399 to 674, all together implying more than 1000-fold increase in efficiency (see [44]), in the price of negligible change in accuracy.
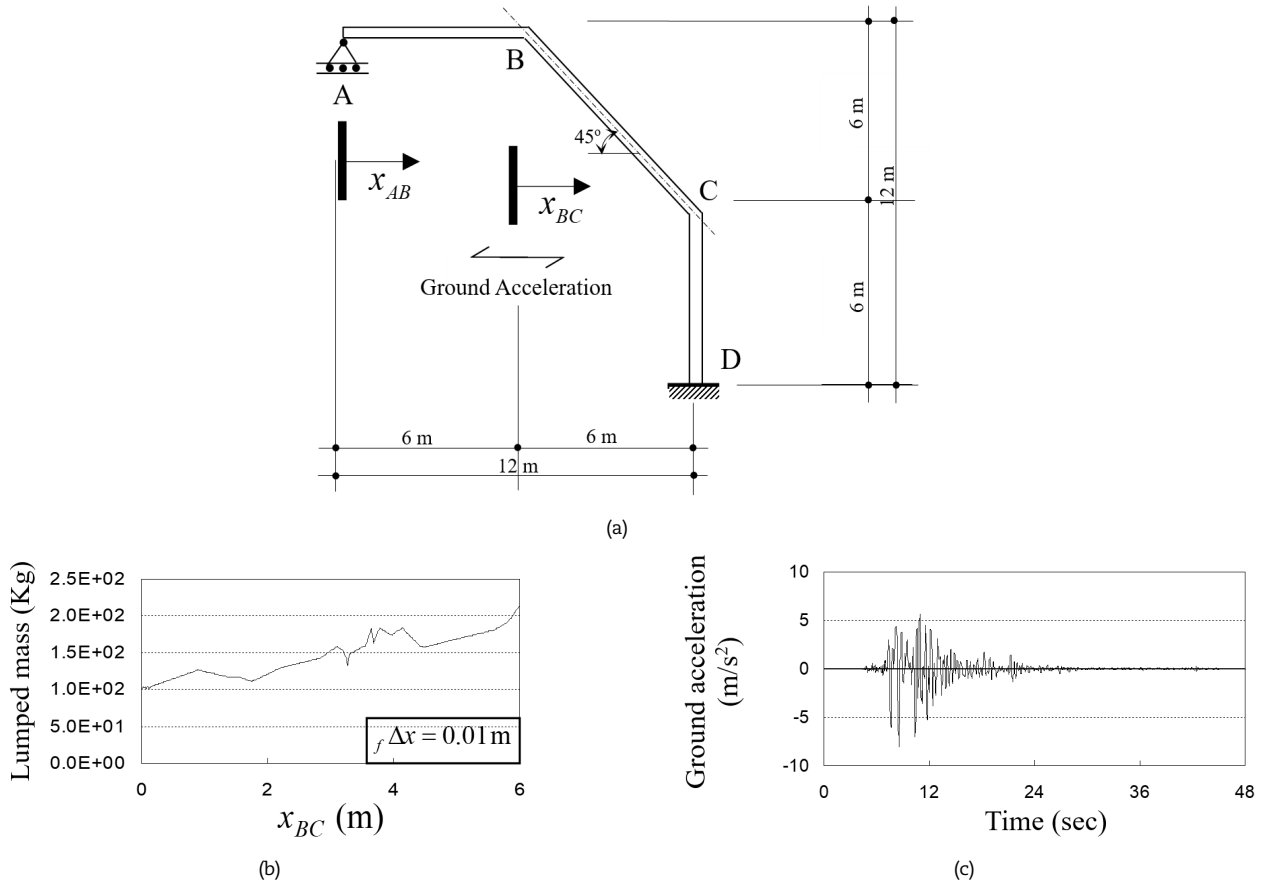

(a)

(b)

(c)

Fig. 2. Structural system in the first example: (a) structural model, (b) mass, (c) ground motion
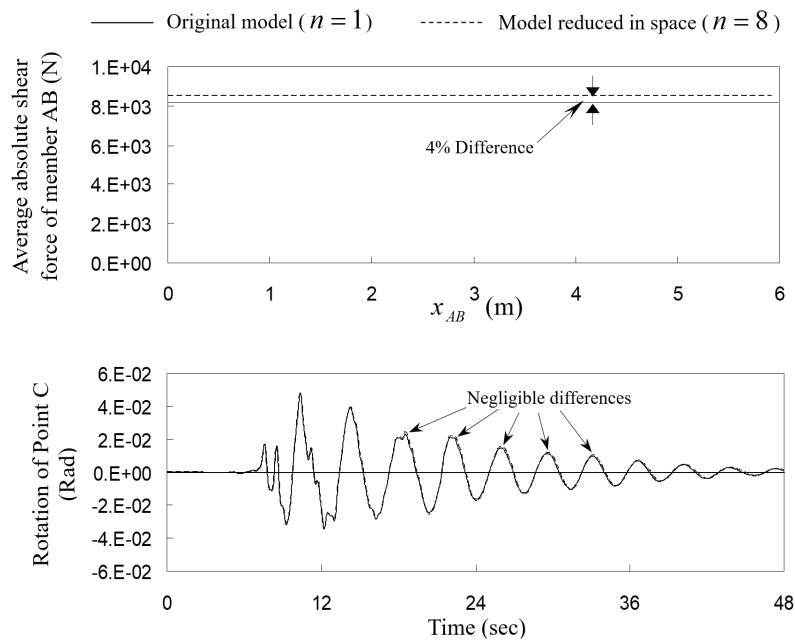

**Fig. 3.** Typical responses computed for the structural system in the first example

Table 3. Complementary information about the structural model in the first example

Member	Profile	Cross sectional Area (m²)	Moment of Inertia (m⁴)	Modulus of Elasticity (N/m²)	Damping
AB					Rayleigh damping considering 5%
BC	IPE 330	0.00626	0.0001177	2.1E11	viscous damping ratio for the 1st
CD					and 3rd natural modes

The study was repeated after replacing the lumped mass with a distributed mass. The results, not reported here for the sake of brevity, were conceptually identical to those for the lumped mass, with regard to both accuracy and efficiency.

3.3 Example of a building structure

Figure 4(a) and Table 4 introduce the building structural system under consideration in this example. The system's main mass is consisted of 9000 Kg lumped masses at the beam-column joints (totally equal to 4.68×10^5 Kg). Besides, lumped masses in a digitized format are considered at a first floor's corner beam, as displayed in Figs. 4(a) and 4(b) (totally, about 3×10^5 Kg). The dynamic excitation is the two-component ground acceleration addressed in Fig. 4(c). The system was first analyzed, using linear two-node beam-column elements [35] (see Table 5) and direct time integration with the HHT (α = -0.2) method [45] and steps equal to the dynamic excitation's digitization-step (i.e. 0.01 sec). The nonlinearity ($P - \Delta$) was modeled, considering modified Newton-Raphson iterations and a tolerance equal to 10^{-6} [37, 44]. To study the performance of the adapted technique, the analysis was then repeated thrice, after considering n = 2, 2.4, 2.5 in implementation of the technique; the element lengths are addressed in Table 5. The resulting responses are typically reported in Fig. 5; and the reductions in run-time and number of degrees of freedom are reported in Table 6. Obviously, the performance is satisfactory. Specifically, for all three values of n, implementation of the proposed adaptation reduced the computational effort (the reductions were in run-time, and in-core memory, and led to more than 16 fold increase in efficiency), while the response accuracy changed negligibly. It is also worth noting that, when n = 2.4, Steps 6 and 7 of the procedure in Section 2.2 were applied; see Table 5. Both from a model reduction point of view, and also when considering the practical aspects, the observations in this example are notable, because of the nonlinearity of the behavior, and the structure's shape topology and building usage; see [46, 47].

(a)

(b)

(c)
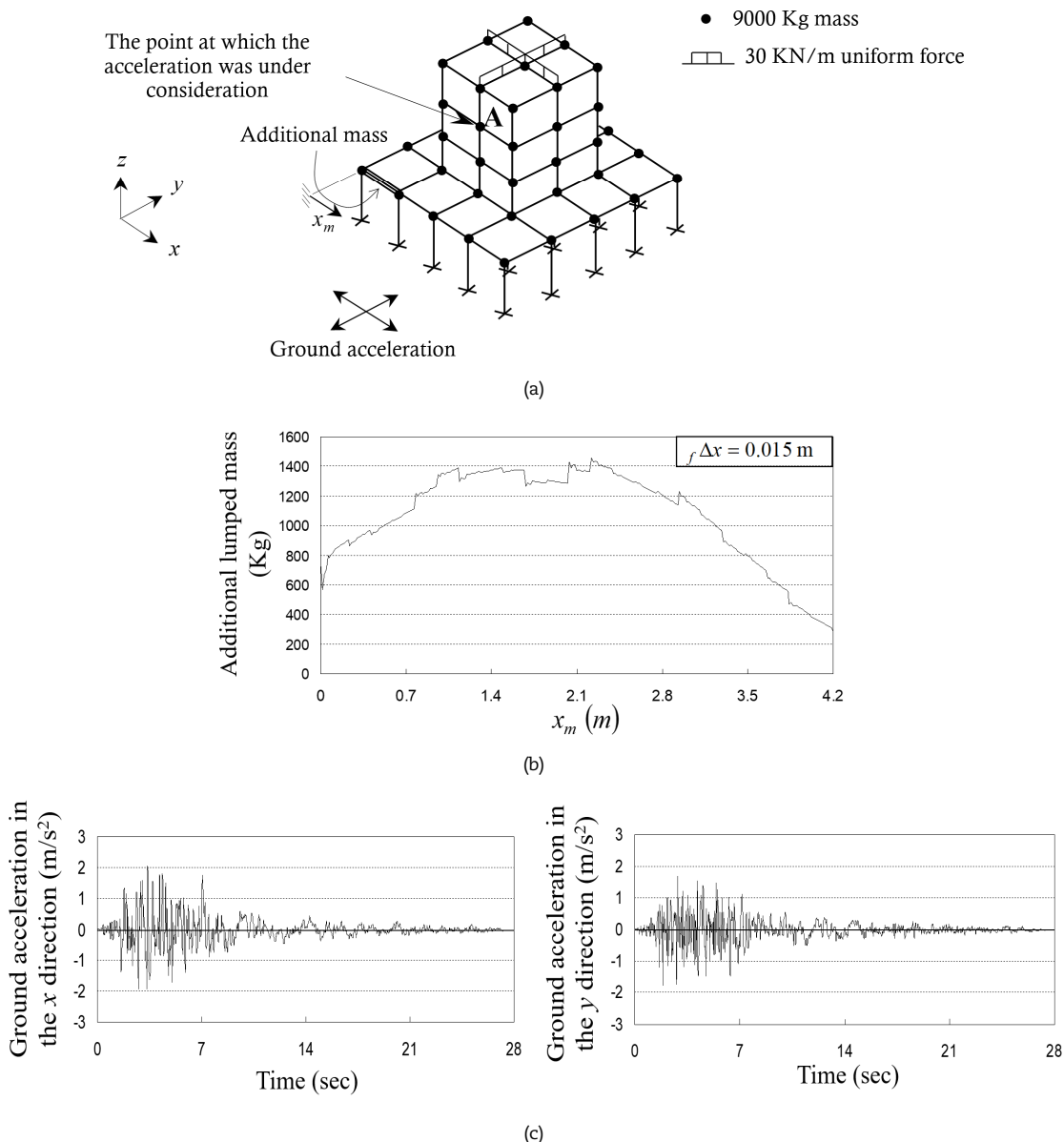
Fig. 4. Structural system in the second example: (a) structural model, (b) mass, (c) ground motion

**Table 4.** Complementary information about the structural model in the second example

Member	Material	Length (m)	Cross section (mm × mm)	Modulus of Elasticity (N/m²)	Damping
Columns in the 1st floor		4.8	Square (450×450)		Rayleigh damping considering 5% viscous damping ratio for the 1st and 5th natural modes
Columns in the 2nd, 3rd, and 4th floors	Reinforced Concrete	3.3	Square (400×400)	2.0E10	
Beams		4.2	Rectangular (250×350)		

Table 5. Lengths of beam-column elements in the finite element model in the second example (m)

n	Beam with mass (m) (see Fig. 4(a))	Beams without mass (m) (see Fig. 4(a))	Columns of the 1st floor (m)	Columns of the 2nd, 3rd, and 4th floors (m)
1.0 (no reduction)	0.015	0.42	0.48	0.33
2.0	0.030	0.84	0.96	0.66
2.4	0.035897435897	0.84	0.96	0.66
2.5	0.0375	1.05	1.20	0.825

Table 6. Run-times and numbers of degrees of freedom in implementation of the proposed adaptation in the second example

n	Run-time (sec)	Number of degrees of freedom
1.0 (no reduction)	2992	8844
2.0	1438	4194
2.4	1422	4056
2.5	1375	3264
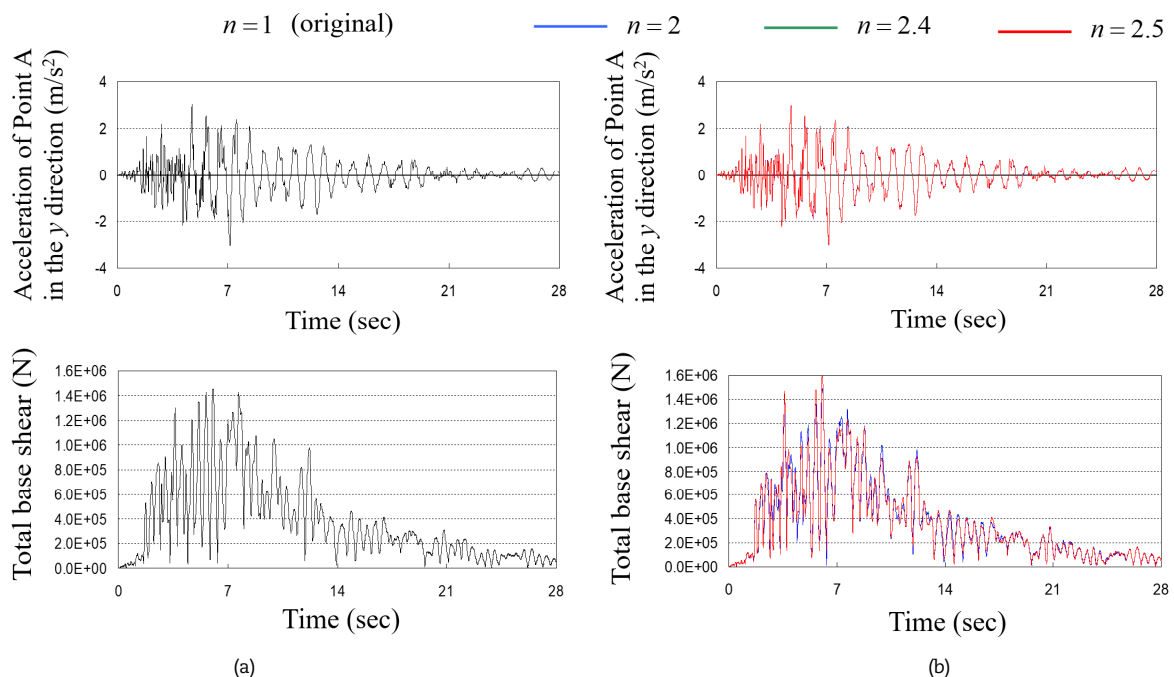
Fig. 5. Typical responses computed for the structural system in the second example: (a) Original model, (b) Reduced models

3.4 Elaborate example

The purpose of this example is to study the performance of the proposed adaptation when implemented in analysis of structures with considerable number of degrees of freedom, non-proportional damping, spread nonlinearity, and realistic mass. Figure 6 and Table 7 introduce the structural system. The mass of the system is consisted of the lumped masses of members AB, BC, and CD, available in digitized format, as addressed in Fig. 6(b) (the profile of the digitized mass was taken from the longitudinal profile of the Brenner Base tunnel, a part of the future TEN No. 5 corridor Helsinki-Valletta [33]). The ground acceleration is as addressed in Fig. 6(c). And, as implied in Fig. 6 and Table 7, the beam-columns behavior is linear-elastic/perfectly-plastic, and the system is non-classically damped [42] (because of the one linear viscous damper in the system); for the importance of these characteristics, see [29, 46, 47].

In analysis of the original model (no reduction), two-node nonlinear beam-column elements [35], sized one centimeter, were used for finite element discretization in space (see Table 8). The resulting semi-discretized equation of motion was analyzed in time, using Bathe's time integration method [48], with steps equal to the earthquake record's digitization step, i.e. 0.02 sec, and the Newton Raphson method for nonlinearity iterations [37, 44] with a tolerance over displacements equal to 10^{-4}. The results are typically reported in Fig. 7(a). The proposed adaptation was then implemented twice, using $n = 5$ and $n = 18$, for changing the mass record, according to the procedure stated in Section 2.2. It is notable that when $n = 18$, Steps 6 and 7 of the procedure were to be applied, and the consequence was different elements' sizes throughout the structure (see Table 8). The resulting responses are reported in Figs. 7(b) and Fig. 7(c), and the computational effort is reviewed in Table 9 (specifically, the decrease in computational effort is more than 8300 fold when $n = 18$; see [44]). These, plus the fact that the mass was totally digitized and associated with a large part of the structure, evidence the possibility of good performance of the proposed adaptation, for problems with moderately large number of degrees of freedom, spread nonlinearity, and non-classical damping.

Towards another evidence for the good performance, the study was repeated after replacing the damping (see Fig. 6(a)), with 2% proportional damping, considering the first and third natural modes. The results were conceptually identical to those presented in Fig. 7 and Table 9, not reported here for the sake of brevity.
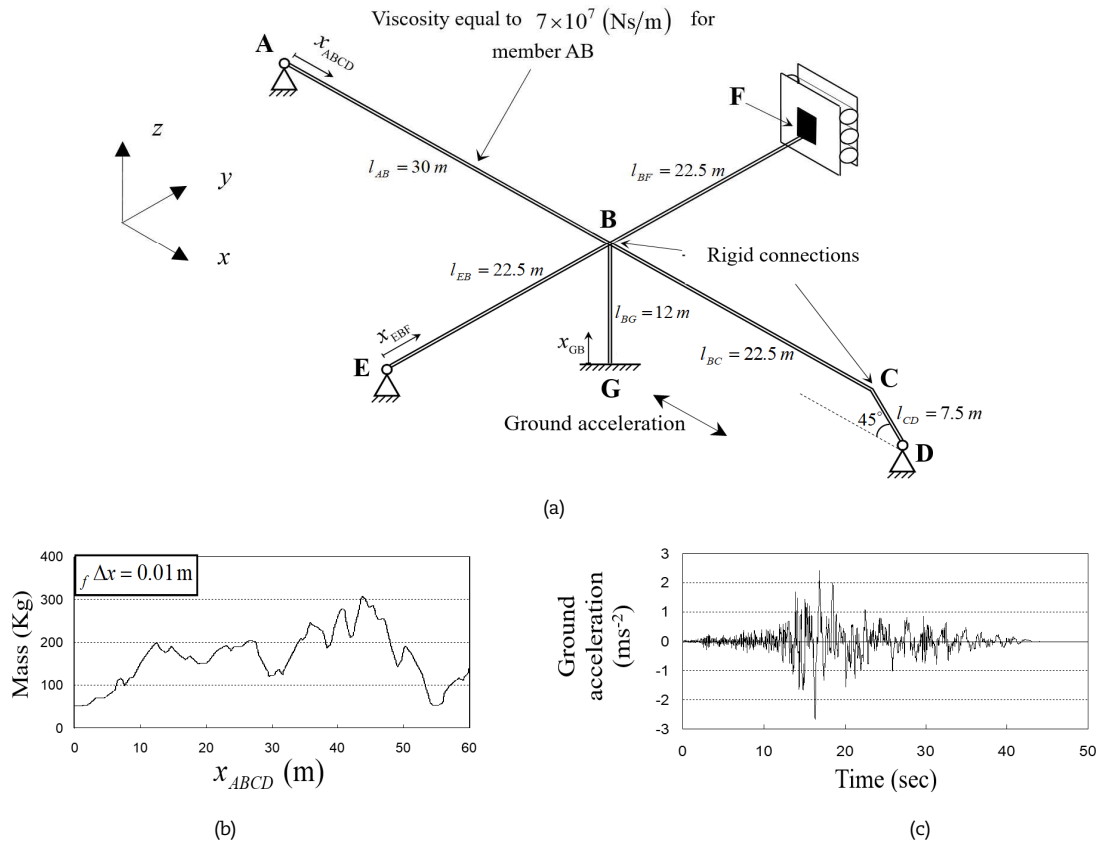
Fig. 6. Structural system in the third example: (a) structural model, (b) mass, (c) ground motion
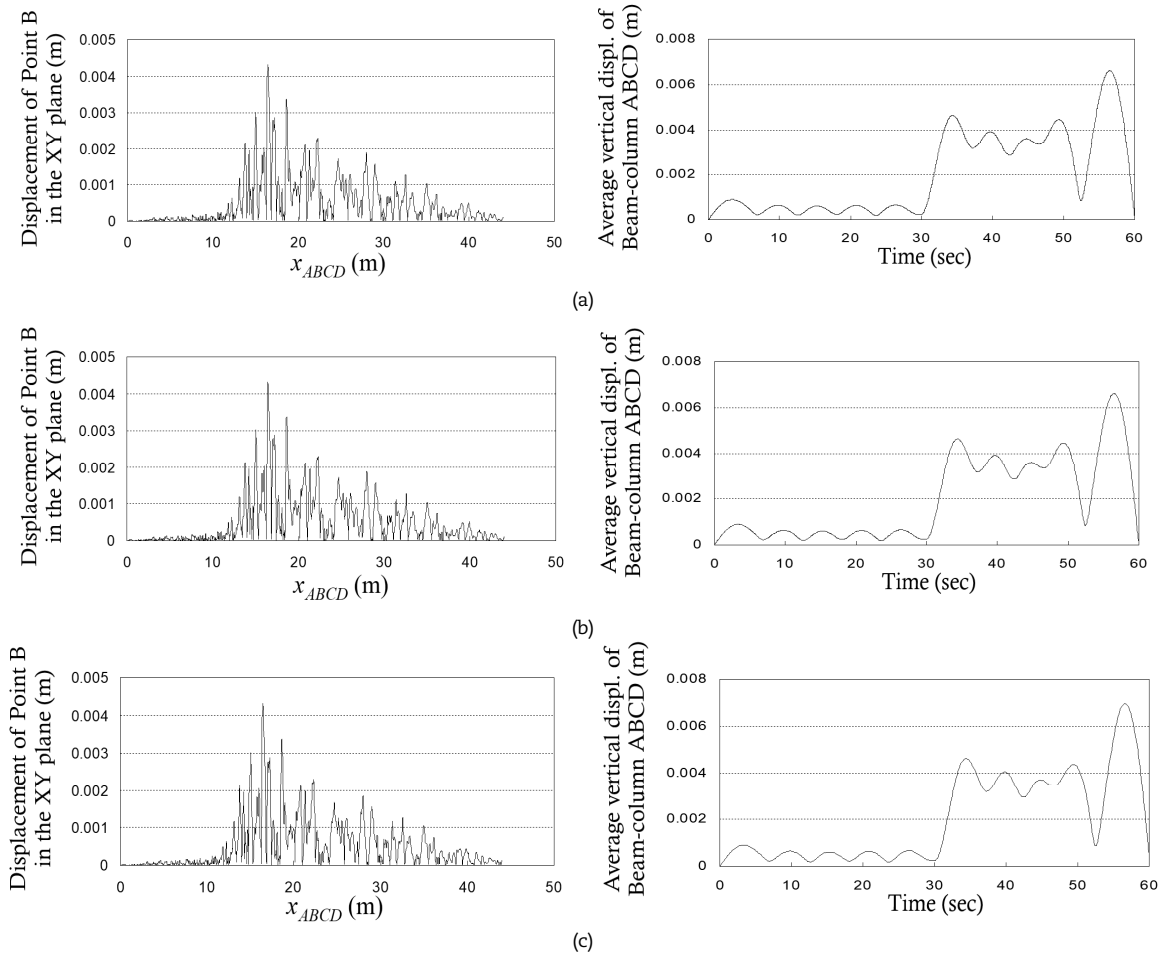

Fig. 7. Typical responses computed for the structural system in the third example: (a) Original model (n = 1), (b) Reduced model (n = 5), (c) Reduced model (n = 18).

Table 7. Complementary information about the structural model in the third example

Beam-column	Profile	Modulus of Elasticity (GPa)	Yield strength (GPa)	Moment of Inertia (cm^4)	Area (cm^2)	Polar Moment of Inertia (cm^4)
AB, BC, CD, BF, EB	IPB 500	210	0.23536	1.072×10^5	239	1.198×10^5
BG	Box 400*400*40	210	0.23536	1.259×10^5	576	2.519×10^5

Table 8. Lengths of beam-column elements in the finite element model in the third example (m)

n	Member AB	Member BC	Member CD	Member EB	Member BF	Member GB
1.0 (no reduction)	0.01	0.01	0.01	0.01	0.01	0.01
5.0	0.05	0.05	0.05	0.05	0.05	0.05
18.0	0.179641	0.18	0.178571	0.18	0.18	0.179104

Table 9. Run-times and numbers of degrees of freedom in implementation of the proposed adaptation in the third example

n	Run-time (sec)	Number of degrees of freedom
1.0 (no reduction)	8190.66 (more than 2 hours)	70186
5.0	1790.38 (about 30 minutes)	14026
18.0	320.512 (about 5.5 minutes)	3892

Table 10. Natural periods of the structural models in Section 3 (in seconds)

Example 1						Example 2						Example 3					
n	Natural mode number					n	Natural mode number					n	Natural mode number				
	1	2	3	4	5		1	2	3	4	5		1	2	3	4	5
1*	3.62	0.97	0.41	0.21	0.13	1*	0.93	0.87	0.81	0.72	0.68	1*	8.93	5.04	3.89	3.13	2.61
						2	0.93	0.87	0.81	0.71	0.65	5	8.93	5.04	3.88	3.13	2.61
8	3.63	0.98	0.42	0.21	0.14	2.4	0.93	0.87	0.81	0.71	0.65						
						2.5	0.93	0.87	0.81	0.71	0.63	18	8.94	5.04	3.89	3.14	2.62

* Modal analysis of the original model (no reduction)

4. Discussion

The main achievement of this paper is replacement of dynamic finite element models of a class of structures with models with less degrees of freedom, using a technique originally proposed in 2008 [3, 15, 18]. From a model reduction point of view, the adapted technique can be easily and cheaply implemented prior to structural analysis by arbitrary software. The resulting reduction in computational effort is considerable and the change of accuracy can be negligible. These are significant features of the proposed adapted technique, that explain further study towards elimination of the main restriction of the technique, i.e. applicability only to analysis of assemblies of beam-columns with masses available in digitized format.

Regarding the accuracy, it is worth noting that, as displayed in Section 3, the proposed adapted technique preserves the accuracy of the response time history, the modal characteristics can be preserved, as well; see Table 10.

In view of the explanations above the proposed adaptation is significant, in practice, and academia, though still limited to a special class of structures. Regarding academic researches, consider that there are many researches in the field of earthquake engineering, based on time history analysis, e.g. IDA (Incremental Dynamic Analysis); see [49-51]. These studies, which are mostly concentrated on structures consisted of beam-columns, are inherently time-consuming. In such studies, when the mass is available totally or partially in digitized format, e.g. fragility analysis of lengthy underground structures, the adaptation proposed in this paper can considerably accelerate the study. Furthermore, with attention to the simple formulation of the adapted technique, and that the proposed adaptation affects only the lumped mass, efficiently teaming of the proposed adaptation with many model reduction and model modification methods (e.g. [29, 30, 52, 53]), as well as adaptive mesh generators (e.g. [24]), and methods accelerating time integration (e.g. [15]), sounds plausible. More efficient model reduction and structural analysis, and more efficient study of the sensitivities (to the model's parameters), can be the practical consequences. Further detailed investigation is essential.

Despite the above-mentioned positive points, some ambiguities need to be clarified. One is the expression "dynamic analysis" in the title of the paper. Time integration is the most powerful broadly accepted tool for transient analysis [42, 54]. Nevertheless, for linear dynamic analyses, modal superposition [9, 42] is a competitive alternative. Replacement of the digitized mass as explained in Section 2 [15, 18, 20, 21] can enhance the efficiency of modal superposition analysis. The natural periods' good accuracy in Table 10, together with the details of modal superposition [42], and the good accuracies reported in Section 3, entail the possibility of responses' good accuracy when modal superposition is used for analysis in time. These imply that, as stated in the title, the proposed adaptation can simplify the dynamic analyses, when the analysis in time is by time integration or modal superposition. Surely, further investigation is essential.

Another ambiguity, that shadows practical implementation of the proposed adapted technique, as well as the predecessor techniques [15, 18, 20, 21], is assignment of appropriate values to n; see Steps 1-4 of the procedure in Section 2.2. Despite the efforts carried out in the past years (e.g. [36]), no established achievement is reported yet.

Finally, disagreement between the notion of lumped mass and the assumption of continuity for the function available in digitized format to be digitized in larger steps; see the appendix [3, 15], is a potential question. Returning to the theoretical details of the technique proposed in 2008 [15], it is sufficient that the digitized data can be represented by a function continuous in the analysis time interval (see [38]). The digitized lumped mass is a discontinuous function that since can be expressed as a summation of Kronecker-delta functions [12, 13], can be with desired accuracy extended to a continuous or even smooth function [38]. Accordingly, the technique [15, 18] can be implemented to replace the lumped mass with a mass digitized in larger steps. The examples in Section 3 were set with lumped masses (rather than distributed masses), specifically in order to display the validity of this claim.

In implementation of the proposed adapted technique, some restrictions exist. The inherent restriction is the type of the structural system, i.e. assemblies of beam–columns. Besides,

1. The mass or a part of the mass shall be available in digitized format (with constant digitization step).
2. The mass shall be represent able as a function continuous with respect to the beam–column's longitudinal axis.

Restriction 1 can be considered satisfied, when the mass is obtained from computation on some digitized information (e.g. soil mass associated with underground tunnels and pipes, depending on the soil properties and depth). In such situations, the instrumentations used for recording the information, e.g. properties of soil and depth, mostly present results in digitized format

[17]. Otherwise, when the mass is not digitized, the restriction can be eliminated by replacing the analytical mass with a digitized representation of a lumped or distributed mass (for adequate accuracy, the digitization step need to be sufficiently small). With regard to Restriction 2, continuity of the mass can generally be accepted, considering that the source of the data is mostly natural phenomena, e.g. ground elevation. The only case this restriction cannot be satisfied is when the mass is distributed and at specific points changes abruptly. This is not only very rare, but also can be eliminated, by using special mathematical functions and accepting the negligible inaccuracy, e.g. see [55].

In view of the achievements, ambiguities, and restrictions above, the research reported in this paper can be continued in the following directions:

1. Testing the achievements in analysis of beam-columns' assemblies with more complicated behavior.
2. Elimination of the restriction on applicability (i.e. the assemblies of beam-columns)
3. Extension of the study to other methods of discretization, e.g. finite difference.
4. Extension of the study to combination of the technique's three successful applications [15, 18, 20, 21], e.g. simultaneous reduction in space and time for assemblies of beam-columns (the first steps are taken; see [56]).
5. Teaming of the proposed adaptation with other methods dedicated to efficient analysis and model modification.
6. Using the achievements to accelerate time-consuming computations, such as IDA and fragility analysis [49-51].
7. Developing a process for assigning an appropriate value to n.
8. Improvement towards more reduction of the computational effort and more response accuracy.

5. Conclusions

By slight modification in the technique proposed in [15, 18], reduction in space, or in other words replacing finite element models with models with less degrees of freedom, is possible in dynamic finite element analysis, when

1. The models represent beam-columns' assemblies with lumped or distributed masses available in digitized format.
2. The digitized masses can be considered smooth with respect to the beam-columns longitudinal axes.

Dynamic analysis of the reduced models can be considerably more efficient than analysis of the original models (more than 8300 fold in the third example), and the change in accuracy can be negligible (see Figs. 3, 5, and 7). Additionally,

1. Implementation of the adapted technique is simple and computationally cheap.
2. The proposed adaptation can be plugged in the preprocessing stage of arbitrary structural analysis software.
3. The good performance of the adapted technique can be anticipated in analysis of simple and complicated linear or nonlinear beam-columns assemblies, damped classically or non-classically, regardless of the method used for integration in time.

Future perspective of the presented study is also briefly addressed.

Author Contributions

A. Soroushian planned the scheme, initiated the project and suggested the experiments; S. Amiri conducted the experiments and analyzed the empirical results; S. Amiri developed the mathematical modeling and examined the theory validation. The manuscript was written through the contribution of both authors. Both authors discussed the results, reviewed and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

References

[1] Henrici, P., *Discrete Variable Methods in Ordinary Differential Equations*, Prentice-Hall, NJ, 1962.
[2] Ralston, A., Rabinowitz, P., *First Course in Numerical Analysis*, McGraw-Hill, USA, 1978.
[3] Soroushian, A., Integration step size and its adequate selection in analysis of structural systems against earthquakes, *Computational Methods in Earthquake Engineering*, 3, 2017, 285-328.
[4] Belytschko, T., Hughes, T.J.R., *Computational Methods for Transient Analysis*, Elsevier, The Netherlands, 1983.
[5] Qu, Z.Q., *Model Order Reduction Techniques with Applications in Finite Element Analysis*, Springer Science & Business Media, 2013.
[6] Hairer, E., Wanner, G., *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems*, Springer, USA, 1996.
[7] Kincaid, D.R., Cheney, E.W., *Numerical Analysis: Mathematics of Scientific Computing*, Vol. 2, American Mathematical Soc., USA, 2002.
[8] Henrych, J., *Finite Models and Methods of Dynamics in Structures*, Elsevier, The Netherlands, 1990.
[9] Argyris, J., Mlejnek, J.P., *Dynamics of Structures*, Elsevier, The Netherlands, 1991.
[10] Wriggers, P., *Computational Contact Mechanics*, John Wiley & Sons, New York, 2002.
[11] NZS 1170, *Structural Design Actions, Part 5: Earthquake Actions-New Zealand*, New Zealand, 2004.
[12] Apostol, T.M., *Calculus*, Vol. I, John Wiley & Sons, New York, 1967.
[13] Greenberg, M.D., *Foundations of Applied Mathematics*, Prentice-Hall, USA, 1978.
[14] Wood, W.L., *Practical Time Stepping Schemes*, Oxford, New York, 1990.

[15] Soroushian, A., A technique for time integration with steps larger than the excitation steps, *Communications in Numerical Methods in Engineering*, 24(12), 2008, 2087–2111.
[16] Penry, S.N., Wood, W.L., Comparison of some single-step methods for the numerical solution of the structural dynamic equation, *International Journal for Numerical Methods in Engineering,* 21(11), 1985, 1941–55.
[17] Havskov, J., Alguacil, G., *Instrumentation in Earthquake Seismology (Modern Approaches in Geophysics)*, Springer, Germany, 2004.
[18] Soroushian, A., Zarabimanesh, Y., Soleymani, K., Khalkhali, S.M., A new technique for fractional enlargement of integration steps in transient analysis against digitized excitations, *In The Proc. International Conference on Structural Engineering Dynamics (ICEDyn 2017)*, Ericeira, Portugal, 2017.
[19] Chen, C.C., Robinson, A.R., Improved time-history analysis for structural dynamics. I: Treatment of rapid variation of excitation and material nonlinearity, *Journal of Engineering Mechanics*, 119(12), 1993, 2496-2513.
[20] Farahani, E.M., Ganji, A., Maalek, S., Soroushian, A., Reduction of computational cost in FEM analysis of beams subjected to digitized static loadings, *In The Proc. 10th National Congress on Civil Engineering (11ICCE)*, Tehran, Iran, 2017 (in Persian).
[21] Soroushian, A., Farahani, E.M., Efficient static analysis of assemblies of beam-columns subjected to continuous loadings available as digitized records, *Frontiers in Built Environment*, 4, 2019, 83.
[22] Idelsohn, S.R., Cardona, A., A reduction method for nonlinear structural dynamic analysis, *Computer Methods in Applied Mechanics and Engineering*, 49(3), 1985, 253-279.
[23] Spiess, H., Wriggers, P., Reduction methods for FE analysis in nonlinear structural dynamics, *PAMM*, 5, 2005, 135–6.
[24] Ramm, E., Rank, E., Rannacher, R., Schweizerhof, K., Stein, E., Wendland, W., Wittum, G., Wriggers, P., Wunderlich, W., *Error-Controlled Adaptive Finite Elements in Solid Mechanics*, John Wiley & Sons, 2003.
[25] Adeli, H., Kamal, O., *Parallel Processing in Structural Engineering*, CRC Press, 2014.
[26] Jung, Y.K., Qu, Z.Q., Jung, D.S., Structural dynamic condensation method with an iterative scheme, *KSCE Journal of Civil Engineering*, 8(2), 2004, 205–211.
[27] Craig, R.R., Substructure Methods in Vibration, Journal of Mechanical Design, 117, 1995, 207–213.
[28] Koutsovasilis, P., Beitelschmidt, M., Model order reduction of finite element models: improved component mode synthesis, *Mathematical and Computer Modelling of Dynamical Systems*, 16(1), 2010, 57–73.
[29] Avitabile, P., Twenty years of structural dynamic modification-a review, *Sound and Vibration,* 37(1), 2003, 14-27.
[30] Nad, M., Structural dynamic modification of vibrating systems, *Applied and Computational Mechanics*, 1, 2007, 203–214.
[31] Besselink, B., Tabak, U., Lutowska, A., Van de Wouw, N., Nijmeijer, H., Rixen, D.J., Hochstenbach, M.E., Schilders, W.H.A., A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control, *Journal of Sound and Vibration*, 332(19), 2013, 4403–22.
[32] Relun, N., Neron, D., Boucard, P.A., A model reduction technique based on the PGD for elastic-visco-plastic computational analysis, *Computational Mechanics*, 51(1), 2013, 83–92.
[33] Bergmeister, K., Life-cycle design for the world's longest tunnel project. In Life-Cycle and Sustainability of Civil Infrastructure Systems: Proceedings of the Third International Symposium on Life-Cycle Civil Engineering (IALCCE'12), Vienna, Austria, October 3-6, 2012 (p. 35). CRC Press, 2012.
[34] Strikwerda, J.C., *Finite Difference Schemes and Partial Differential Equations*, Wadsworth & Books/Cole, Pacific Grove, CA, 1989.
[35] Mazzoni, S., McKenna, F., Scott, M.H., Fenves, G.L. *OpenSees command language manual*, Pacific Earthquake Engineering Research (PEER) Center, 2006.
[36] Soroushian, A., A new approach for practical time integration analysis, *In The Proc. 6th ECCOMAS Thematic Conference on Computational Methods in Structural Dynamics and Earthquake Engineering (COMPDYN 2017)*, Rhodes Island, Greece, 2017.
[37] Allgower, E.L., Georg, K., *Numerical Continuation Methods, An Introduction*, Springer, New York, 1980.
[38] Soroushian, A., Wriggers, P., Farjoodi, J., Asymptotic upper bounds for the errors of Richardson extrapolation with practical application in approximate computations, *International Journal for Numerical Methods in Engineering*, 80(5), 2009, 565–95.
[39] Bathe, K., *Finite Element Procedures*, Prentice-Hall, USA, 1996.
[40] Zienkiewicz, O., Taylor, R., Zhu, J., Nithiarasu, P., *The Finite Element Method*, Elsevier, The Netherlands, 2005.
[41] Soroushian, A., Wriggers, P., Farjoodi, J., Practical integration of semi-discretized nonlinear equations of motion: proper convergence for systems with piecewise linear behavior, *Journal of Engineering Mechanics*, 139(2), 2013, 114–45.
[42] Chopra, A.K., *Dynamics of structures: theory and application to earthquake engineering*, Prentice-Hall, USA, 1995.
[43] Newmark, N.M., A method of computation for structural dynamics, *Journal of Engineering Mechanics*, 85(3), 1959, 67–94.
[44] Cook, R.D., Malkus, D.S., Plesha, M.E., Witt, R.J., *Concepts and Applications of Finite Element Analysis*, Wiley, USA, 2002.
[45] Hilber, H.M., Hughes, T.J., Taylor, R.L., Improved numerical dissipation for time integration algorithms in structural dynamics, *Earthquake Engineering & Structural Dynamics*, 5(3), 1977, 283–92.
[46] Segalman, D.J., Model reduction of systems with localized nonlinearities, *Journal of Computational and Nonlinear Dynamics*, 2(3), 2007, 249–66.
[47] Butlin, T., Woodhouse, J., Champneys, A.R., The landscape of nonlinear structural dynamics: an introduction, *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 373(2051), 2015.
[48] Bathe, K.J., Conserving energy and momentum in nonlinear dynamics: a simple implicit time integration scheme, *Computers & Structures*, 85(7–8), 2007, 437–45.
[49] Vamvatsikos, D., Cornell, C.A., Incremental dynamic analysis, *Earthquake Engineering & Structural Dynamics*, 31(3), 2002, 491–514.
[50] Erberik, M.A., Elnashai, A.S., Fragility analysis of flat-slab structures, *Engineering Structures*, 26(7), 2004, 937–48.
[51] Fragiadakis, M., Vamvatsikos, D., Fast performance uncertainty estimation via pushover and approximate IDA, *Earthquake Engineering & Structural Dynamics*, 39(6), 2010, 683-703.
[52] Herrada, F.J., García-Martínez, J., Fraile, A., Hermanns, L.K.H., Montáns, F.J., A method for performing efficient parametric dynamic analyses in large finite element models undergoing structural modifications, *Engineering Structures*, 131, 2017, 625–38.
[53] García-Martínez, J., Herrada, F.J., Hermanns, L.K.H., Fraile, A., Montáns, F.J., Accelerating parametric studies in computational dynamics: Selective modal re-orthogonalization versus model order reduction methods, *Advances in Engineering Software*, 108, 2017, 24–36.
[54] Chung, J., Hulbert, G.M., A family of single-step Houbolt time integration algorithms for structural dynamics, *Computer Methods in Applied Mechanics and Engineering*, 118, 1994, 1-11.
[55] Mostaghel, N., Davis, T., Representations of coulomb friction for dynamic analysis, *Earthquake Engineering and Structural Dynamics*, 26, 1997, 541–48.
[56] Soroushian, A., Amiri, S., Simultaneous reduction in time and space for dynamic finite element models of beam-column assemblies, *Iranian Journal of Science and Technology, Transactions of Civil Engineering*, 2020, DOI: 10.1007/s40996020-00377-x.

Appendix

The purpose of this appendix is to review and summarize the main formulation proposed in [15, 18]. Given data digitized in constant steps, the formulation presented in [15, 18] can reduce the data (the number of the data), by enlarging the digitization step, using an enlargement ratio larger than one. The presentation below, first considers integer enlargement ratios [15], and then extends the formulation to real enlargement ratios [18].

Consider the positive integer number n (including one) as the enlargement ratio, $\mathbf{f}$ as the vector of the digitized data, and $_f\Delta x$ as the digitization step, where x implies the independent continuously changing variable defining the digitization. Note that the x in the appendix is different from the x in the main body of the paper. For an n-fold reduction of the data, the digitization step $_f\Delta x$ changes to $_f\Delta\tilde{x}$ as (unless stated the n in eq. (A1) is an integer):

$$_f\Delta\tilde{x} = n \, _f\Delta x \tag{A1}$$

and the new vector $\tilde{\mathbf{f}}$ is defined as

$$\tilde{\mathbf{f}}_i = \tilde{\mathbf{f}}(x_i) = \mathbf{f}(x_i) \qquad\qquad\qquad\qquad\qquad \text{when} \quad n = 1$$

$$\tilde{\mathbf{f}}_i = \tilde{\mathbf{f}}(x_i) = \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{when} \quad n = 2,3,4,\cdots$$

$$\begin{cases} \mathbf{g}(x_i) & x_i = 0 \\ \dfrac{1}{2}\mathbf{g}(x_i) + \dfrac{1}{4n'}\sum_{k=1}^{n'}\left[\mathbf{g}(x_i + k_f\Delta x) + \mathbf{g}(x_i - k_f\Delta x)\right] & x_i = n_f\Delta x,\ 2n_f\Delta x,\ \ldots\ x'_{end} - n_f\Delta x \\ \mathbf{g}(x_i) & x_i = x'_{end} \end{cases} \tag{A2}$$

In eq. (A2), n' is defined as

$$n' = \begin{cases} n-1 & \text{when } x_i = n_f\Delta x \\ \begin{cases} \dfrac{n}{2} & n = 2j, \qquad j \in Z^+ \\ \dfrac{n-1}{2} & n = 2j+1, \quad j \in Z^+ \end{cases} & \text{when } x_i = 2n_f\Delta x, 3n_f\Delta x, \ldots, x'_{end} - 2n_f\Delta x \\ n-1 & \text{when } x_i = x'_{end} - n_f\Delta x \end{cases} \tag{A3}$$

x'_{end} is the smallest value not smaller than x_{end} that is also an integer multiplier of $n_f\Delta x$, and $\mathbf{g}(x_i)$ is defined as

$$\mathbf{g}(x_i) = \begin{cases} \mathbf{f}(x_i) & \text{when } 0 \le x_i \le x_{end} \\ 0 & \text{when } x_{end} < x_i < x'_{end} \end{cases} \tag{A4}$$

Equations (A1-A4) clearly define the computation of $\tilde{\mathbf{f}}$, in terms of $\mathbf{f}$ and n, when n is a positive integer. The remainder of this appendix is dedicated to the extension of the enlargement ratios to positive real numbers n, i.e.

$$n = r > 1, \qquad r \in R^+ \quad (R^+ \text{ is the set of positive real numbers}) \tag{A5}$$

From simple calculus [12], for an arbitrary positive real number n larger than one, there always exist two sequential integer numbers, n_1 and n_2, such that:

$$\begin{aligned} n_1 &< n \le n_2 \\ n_2 &= n_1 + 1 \\ n_1, n_2 &\in Z^+ \end{aligned} \tag{A6}$$

By considering the computation addressed in eqs. (A1-A4) for the two integers n_1 and n_2, we will obtain $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$ corresponding to n_1 and n_2, respectively. The associated digitization steps are respectively equal to:

$$\begin{aligned} {}_f\Delta\tilde{x}_1 &= n_1\ {}_f\Delta x \\ {}_f\Delta\tilde{x}_2 &= n_2\ {}_f\Delta x \end{aligned} \tag{A7}$$

Concentrating on $\tilde{\mathbf{f}}_1$, by using linear interpolation, the digitization step of $\tilde{\mathbf{f}}_1$ can be changed to ${}_f\Delta\tilde{x}$, where ${}_f\Delta\tilde{x}$ is obtainable from eq. (A1) considering non-integer values for n. The last computation can be repeated for $\tilde{\mathbf{f}}_2$. Accordingly, $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$ are now both digitized data, with the digitization steps similarly equal to ${}_f\Delta\tilde{x}$. In this stage, the two digitized data, i.e. $\tilde{\mathbf{f}}_1$ and $\tilde{\mathbf{f}}_2$, though correspond to different enlargement ratios, i.e. n_1 and n_2, are both digitized in steps ${}_f\Delta\tilde{x}$. Point-by-point linear interpolation between the members of the two data, as

$$\tilde{\mathbf{f}} = (n_2 - n)\tilde{\mathbf{f}}_1 + (n - n_1)\tilde{\mathbf{f}}_2 \tag{A8}$$

entails the reduced data corresponding to n [18], when n is a real number larger than one. As a validity check, when n approaches n_1 or n_2, $\tilde{\mathbf{f}}$ approaches $\tilde{\mathbf{f}}_1$ or $\tilde{\mathbf{f}}_2$, respectively.

ORCID iD

Aram Soroushian https://orcid.org/0000-0001-5047-5969
Saeed Amiri https://orcid.org/0000-0002-0077-4298

Publisher's Note Shahid Chamran University of Ahvaz remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.