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Research Paper

Numerical Solution of the Time Fractional Reaction-advection-diffusion Equation in Porous Media

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Abstract. In this work, we obtained the numerical solution for the system of nonlinear time-fractional order advection-reaction-diffusion equation using the homotopy perturbation method using Laplace transform method with fractional order derivatives in Liouville-Caputo sense. The solution obtained is very useful and significant to analyze many physical phenomenons. The present technique demonstrates the coupling of homotopy perturbation method and the Laplace transform technique using He's polynomials, which can be applied to numerous coupled systems of nonlinear fractional models to find the approximate numerical solutions. The salient features of the present work is the graphical presentations of the numerical solution of the concerned nonlinear coupled equation for several particular cases and showcasing the effect of reaction terms on the nature of solute concentration of the considered mathematical model for different particular cases. To validate the reliability, efficiency and accuracy of the proposed efficient scheme, a comparison of numerical solutions and exact solution are reported for Burgers' coupled equations and other particular cases of concerned nonlinear coupled systems. Here we find high consistency and compatibility between exact and numerical solution to a high accuracy. Presentation of absolute errors for given examples are reported in tabulated and graphical forms that ensure the convergence rate of the numerical scheme.

Keywords: Fractional Calculus, Homotopy Perturbation, He's Polynomials, Sub-diffusion, Porous Media.

1. Introduction

The theory fractional calculus is as old as ordinary calculus. Fractional order derivatives and integral are the more general case of integer order derivatives and integral and can be defined by replacing the integer order by fractional order. Numerous physical models in recent studies in engineering and science are being modeled by the means of fractional derivatives. In the last few decades, fractional derivatives have been widely studied by researchers in many important aspects viz., chemical [1], signal theory [2, 3], fluid [4, 5] etc. Finding the exact solution of numerous nonlinear fractional derivatives model is still a tough task for researchers, therefore numerical solution of these fractional models are of significant interest. The applications of fractional calculus can be successfully modeled by different types fractional differential equations (FDEs) like linear or non-linear. The applications of different kinds of FDEs in mathematical model are increasing due to extensive applications of fractional calculus and a number of numerical schemes for their solutions. It has helped fractional calculus to become a very significant and attractive research field.

Several techniques have been discovered which provide the approximate numerical solution for fractional order ordinary differential equations (ODEs) and partial differential equations (PDEs) viz., finite difference method (FDM)[6, 7], predictor-corrector method, fractional differential transform [8], Adomain decomposition [9], eigenvector expansion, generalized block pulse operational matrix method [10] and the method given in [42-48] etc. Some numerical methods based upon operational matrices of fractional differentiation and integration with Legendre wavelets [11], Gegenbauer wavelet [12], Chebyshev wavelets [13-15], sine wavelets [16], Haar wavelets [17] have been developed for finding the numerical solutions of fractional order differential equations, integro-differential equations. Here one can notice that several physical phenomenas can be modeled by using FPDEs and their systems[18, 19,41]. Coupling of various physical phenomenons by some mathematical model is global in applied sciences.

Commonly, it is known that diffusion process is a physical process where a substance move from a high concentration zone to a low concentration zone, the "Diffusion" is taken from the Latin word *diffundere* which means spreading of molecule from high to low concentration area. The process of diffusion mainly occurs in gases or liquid because their molecules randomly moves from one place to other. Diffusion type equation is a kind of PDE which portrays density profile dynamics in the process of diffusion of a substance [20]. Approximate numerical solution of the reaction-diffusion equation has become a center of wide research due to its nature and have been applied in modeling of oil reservoir simulations, flow of fluids in porous media, global water production

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and many numerous natural phenomenon. Fractional reaction-diffusion equations (FRDEs) are widely used in demonstrating abnormal slowly diffusion process for non-conservative system viz., the FRDEs are the mathematical modeling of the change in time and space of the concentration of one or more molecules of a substance.

In the study of nonlinear physical problems, fractional order reaction-diffusion model plays a significant role, therefore authors are motivated to find the approximate numerical solution of space-time fractional order reaction-diffusion equation and to analyses their physical behavior in porous media associated with Liouville-Caputo fractional order derivatives. In the present scientific contribution, the author analyzed the mathematical form of a physical problem. Here author traces the behavior of two interacting solute species under the influence of reaction, diffusion, and advection in porous media. During the fluid flow, there is no. of affecting factors viz., nonlinearity, transient nature, and coupling of mathematical models that produce the complexity to tackle the physical problem. In the porous medium, two solute species interact, and the phenomenon of diffusion occurs. Our main goal is to find the long-term behavior of the considered nonlinear system. Here we present a fractional nonlinear mathematical model of such a physical problem containing two solute variables. In this paper, author consider the fractional porous media equation as

$$\left.\begin{aligned}\frac{\partial^{\alpha_1}u(x,t)}{\partial t^{\alpha_1}}&=D_1\frac{\partial^2 u(x,t)}{\partial x^2}+B_1\frac{\partial^{\beta_1}v(x,t)}{\partial x^{\beta_1}}-\gamma_1\frac{\partial u(x,t)}{\partial x}u^m(x,t)+C_1\frac{\partial(v(x,t)u(x,t))}{\partial x}+k_1u(x,t)(1-u(x,t))+f(x,t),\\ \frac{\partial^{\alpha_2}v(x,t)}{\partial t^{\alpha_2}}&=D_2\frac{\partial^2 v(x,t)}{\partial x^2}+B_2\frac{\partial^{\beta_2}u(x,t)}{\partial x^{\beta_2}}-\gamma_2\frac{\partial v(x,t)}{\partial x}v^n(x,t)+C_2\frac{\partial(v(x,t)u(x,t))}{\partial x}+k_2v(x,t)(1-v(x,t))+g(x,t),\end{aligned}\right\} \tag{1}$$

where $0\le x\le 1$, $0<\alpha_1,\alpha_2\le 1$, $0\le t\le 1$, $1<\beta_1,\beta_2\le 2$, with initial conditions

$$v(x,0)=v_0(x),\quad u(x,0)=u_0(x). \tag{2}$$

In the above expressions, D_1,D_2 denotes the diffusion coefficient, γ_1,γ_2 be the advection coefficients, k_1,k_2 denotes the reaction coefficient, B_i's,C_i's are constants and $g(x,t),f(x,t)$ represents the force terms..

The authors of [21, 22] discussed the some particular cases of our concerned model. By putting some appropriate value to the arbitrary constants, equation (1) is reduces to the famous Burgers' equations. The Cole-Hopf transformation is the most common approach to encounter Burgers' equations [21, 23]. The role of nonlinear FPDEs viz., Burgers' equation, Huxley equation, Fisher equation is very important to understand the fluid flow through a porous medium.

For the living things water pollution is a big problem, the FRDEs provides an important water quality mathematical model in environmental science. Ground water contamination, e.g., wastage of man made products directly pollutes the ground water through porous medium and can have serious effects on health and also harms the wildlife. The authors of [24], discussed the importance of mathematical models of ground water resources management. Many scientists and engineers are studying the contamination of the environmental subsurface, which have excellent effect on the progress of research in process of solute transport in the porous media [25-27].

If ground water is contaminated overall, then the rehabilitation is deemed to be too difficult and expensive. Solute transport through the groundwater is topic encountered in the interdisciplinary branch of science and engineering, called hydrology. The following equation represent solute transport in aquifiers

$$\frac{\partial u(x,t)}{\partial t}=d\frac{\partial^2 u}{\partial x^2}-v\frac{\partial u}{\partial x}. \tag{3}$$

Here, $u(x,t)$ is solute concentration, $v>0$, represents average velocity of fluid and d represents the coefficient of dispersion. The above equation is also called advection dispersion equation. This equation also describes probability density function in a continuum for location of particles . The equation can be used in the groundwater hydrology in which the movement of the passive tracers is picked by the flow of the fluid in the porous media.

In this article, we studies new iterative Laplace transform technique to find the approximate numerical solution of considered nonlinear coupled system space-time fractional advection-reaction-diffusion equation. This particular mehtod is an authenticate extension from the the method to find the approximate analytical solution for differential equation[28] to the coupled system of nonliner fractional PDEs. This new iterative Laplace transform technique is basically a coupling of three methods of Laplace transform technique, He's polynomials and homotopy perturbation technique. Integral transform methods viz., Hilbert, Fourier, Stieltjes or Laplace transforms have very significant role to study the ODEs and PDEs. By use of the Laplace transform, ODEs and PDEs system have been transforms to algebraic systems. He's polynomials provides a refined series solution which is primarily due to Ghorbani [29] and Ghorbani and Nadjafi [30]. Numerous researchers have been continuously worked on the homotopy method over many years [31, 32]. Homotopy perturbation method (HPM)has been shown to solve accurately, easily and effectively a wide range of nonlinear physical problems and in general, iterations converges to exact solution rapidly after one or two iterations. A brief discussion and analysis of HPM method are given in this article.

The paper is organized in following ways. Some fundamental concepts for Laplace transform and fractional derivatives are given. In section 3, Homotopy perturbation and He's polynomials are discussed. In Section 4, homotopy perturbation Laplace transform method is utilized for fractional order derivative to solve our proposed model. Few example are given in section 5 to validate the efficiency of proposed scheme. Numerical simulation of model is given in section 6. Achievements of this scientific research is discussed in Conclusion.

2. Basic definitions of Laplace transform and fractional derivatives

2.1 Fractional Derivative

We have given some basic definitions and some notations of the Laplace transform and fractional derivatives which are important for establishing the results in the article.

Definition 1.

The fractional order derivative operator D^{α} of the given order $\alpha>0$ in the Liouville-Caputo sense is given by [33, 34]

$$(D^{\alpha}f)(t)=\begin{cases}\dfrac{d^{z}f(t)}{dx^{z}}, & if\ \ \alpha=z\in N,\\ \dfrac{1}{\Gamma(z-\alpha)}\displaystyle\int_0^t(t-\rho)^{z-\alpha-1}f^{(z)}(\rho)\ \ d\rho, & if\ \ z-1<\alpha<z.\end{cases} \tag{4}$$

Here z is integer. Some properties of Liouville-Caputo derivatives are given as:

$$D^{\alpha}K=0, \tag{5}$$

where K is an arbitrary constant.

$$D^{\alpha}t^{l}=\begin{cases}0, & if\ \ l\in N\cup\{0\},l<\lceil\alpha\rceil,\\ \dfrac{\Gamma(l+1)}{\Gamma(l-\alpha+1)}t^{l-\alpha}, & if\ \ l\in N\cup\{0\},l\geq\lceil\alpha\rceil\quad or\quad l\notin N,l>\lfloor\alpha\rfloor,\end{cases} \tag{6}$$

Here $\lfloor\alpha\rfloor$ and $\lceil\alpha\rceil$ are floor and ceiling function respectively.

2.2 Laplace Transform

We know that Laplace transformation project the considered time fractional reaction diffusion equation from time domain to Laplacian domain. Let $u(t)$ be the usual time domain function of the variable $t\geq 0$ and $\bar{u}(s)$ is the image of $u(t)$ under the Laplace operator in the Laplacian domain. The definition of Laplace transform can be given as

$$\bar{u}(s)=£[u(t)]=\int_0^{\infty}u(t).e^{-st}dt. \tag{7}$$

In the above expression s is the Laplacian parameter. The Laplace transformation of the Liouville-Caputo fractional order derivative is [35]

$$£[\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}]=s^{\alpha}\bar{u}(x,s)-s^{\alpha-1}u_0(x),\quad 0\leq\alpha\leq 1,t\geq 0. \tag{8}$$

3. Homotopy Perturbation Theory and Basic of He's Polynomials

3.1 Homotopy Perturbation Method

The homotopy perturbation method (HPM) is a coupling of the homotopy concept and classical perturbation technique as applied in topology. To brief analysis of HPM for the numerical solution of nonlinear physical problems, we consider following a nonlinear system of PDE

$$\left.\begin{aligned}F(u,v)-f(x,t)=0,x,t\in\Omega=[0,1]\times[0,1],\\ F'(u,v)-f'(x,t)=0,x,t\in\Omega=[0,1]\times[0,1],\end{aligned}\right\} \tag{9}$$

subject to the boundary conditions

$$\begin{aligned}P(u,\frac{\partial u}{\partial\eta})=0,\\ P'(v,\frac{\partial v}{\partial\eta})=0,\end{aligned} \tag{10}$$

where F and F' is a general fractional partial differential operator, P and P' is a boundary operator, $f(x,t)$ and $f'(x,t)$ is a known analytic function, Γ is boundary of Ω and $\partial u/\partial\eta$, $\partial v/\partial\eta$ denotes the derivatives along the normal drawn from outwards of the Ω.

Above mentioned fractional partial differential operator can be further divided in to four parts viz., linear parts $A(u,v)$, $A'(u,v)$ and nonlinear parts $B(u,v)$, $B'(u,v)$ therefore equation (9) become

$$\left.\begin{aligned}A(u,v)+B(u,v)-f(x,t)=0,\\ A'(u,v)+B'(u,v)-f'(x,t)=0.\end{aligned}\right\} \tag{11}$$

By applying homotopy technique, we present the homotopy $u'(x,t,\rho),v'(x,t,\rho)\in\Omega\times[0,1]$ with embedding parameter $\rho\in[0,1]$ as

$$\left.\begin{aligned}H(u',\rho)=(1-\rho)[A(u',v')-A(u_0,v_0)]+\rho[F(u,v)-f(x,t)]=0,\\ H'(v',\rho)=(1-\rho)[A'(u',v')-A'(u_0,v_0)]+\rho[F'(u,v)-f'(x,t)]=0,\end{aligned}\right\} \tag{12}$$

above equation can be transformed as

$$\left.\begin{aligned}H(u',\rho)=A(u',v')-A(u_0,v_0)+\rho A(u_0,v_0)+\rho[B(u',v')-f(x,t)]=0,\\ H'(v',\rho)=A'(u',v')-A'(u_0,v_0)+\rho A'(u_0,v_0)+\rho[B'(u',v')-f'(x,t)]=0.\end{aligned}\right\} \tag{13}$$

From the equation (12), it follows that

$$\left.\begin{aligned} H(u',0) &= A(u',v') - A(u_0,v_0),\\ H'(v',0) &= A'(u',v') - A'(u_0,v_0),\end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned} H(u',1) &= F(u,v) - f(x,t),\\ H'(v',1) &= F'(u,v) - f'(x,t).\end{aligned}\right\} \tag{15}$$

Thus the process of varying ρ from zero to one is same that of $u_0(x,t)$ to $u(x,t)$ and $v_0(x,t)$ to $v(x,t)$. This process is called deformation in topology and $A(u',v') - A(u_0,v_0), F(u,v) - f(x,t)$ and $A'(u',v') - A'(u_0,v_0), F'(u,v) - f'(x,t)$ are called homotopic. Assuming the solution of (12) or (13) in a power series of ρ and embedding parameter ρ as small quantity:

$$\left.\begin{aligned} u' &= u_0' + \rho u_1' + \rho^2 u_2' + \cdots\\ v' &= v_0' + \rho v_1' + \rho^2 v_2' + \cdots\end{aligned}\right\} \tag{16}$$

Choosing $\rho = 1$, we get the approximate numerical solution of (9):

$$\left.\begin{aligned} u &= \lim_{\rho\to 1} u' = u_0 + u_1 + u_2 + \cdots\\ v &= \lim_{\rho\to 1} v' = v_0 + v_1 + v_2 + \cdots\end{aligned}\right\} \tag{17}$$

The coupling of homotopy method and perturbation method is called as the homotopy perturbation method, which has aborted the limitations of usual perturbation technique. The author of [36] has been proved the convergence of the series (17).

3.2 He's Polynomials

Ghorbani [29] has provided the following definitions and properties of He's polynomials.

Definition: The He's polynomial is defined as follows [29]:

$$H_k(v_0, v_1, v_2, \cdots, v_k) = \frac{1}{k!}\frac{\partial^k}{\partial \rho^k} B\Big(\sum_{j=0}^{\infty} \rho^j v_j\Big), \quad k = 0,1,2,\cdots \tag{18}$$

Theorem.1: Consider $B(v)$ is a nonlinear function such that $v = \sum_{j=0}^{k} \rho^j v_j$ where ρ is embedding parameter then we have

$$\Big[\frac{\partial^k}{\partial \rho^k} B(v(\rho))\Big]_{\rho=0} = \Big[\frac{\partial^k}{\partial \rho^k} B\Big(\sum_{j=0}^{\infty} \rho^j v_j\Big)\Big]_{\rho=0} = \Big[\frac{\partial^k}{\partial \rho^k} B\Big(\sum_{j=0}^{k} \rho^j v_j\Big)\Big]_{\rho=0}. \tag{19}$$

Proof. See [29].

4. Application on Considered Model

In this section of the article, iterative Laplace scheme is applied to concerned coupled system of fractional advection-reaction-diffusion equation using HPM and He's polynomials. In general our considered model become

$$\left.\begin{aligned} \frac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}} + M_1 u(x,t) + M_1' v(x,t) + N_1 v(x,t) &= f(x,t),\\ \frac{\partial^{\alpha_2} v}{\partial t^{\alpha_2}} + M_2 v(x,t) + M_2' u(x,t) + N_2 u(x,t) &= g(x,t),\end{aligned}\right\} \tag{20}$$

with initial conditions

$$v(x,0) = v_0(x), \quad u(x,0) = u_0(x). \tag{21}$$

In the above equation, fractional derivative is in Liouville-Caputo sense, M_1, M_2, M_1', M_2' denotes the general linear operator, N_1, N_2 stands for general nonlinear differential operator and $f(x,t), g(x,t)$ represents the force terms.
Operating the Laplace transformation on the equation (20), we get

$$\left.\begin{aligned} £\Big[\frac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}}\Big] + £[M_1 u(x,t) + M_1' v(x,t)] + £[N_1 v(x,t)] &= £[f(x,t)],\\ £\Big[\frac{\partial^{\alpha_2} v}{\partial t^{\alpha_2}}\Big] + £[M_2 v(x,t) + M_2' u(x,t)] + £[N_2 u(x,t)] &= £[g(x,t)].\end{aligned}\right\} \tag{22}$$

Using equation (8) and properties of Laplace transform, we get

$$\left.\begin{aligned} \bar{u}(x,s)s^{\alpha} &= s^{\alpha-1}u_0(x) + £[f(x,t)] - \{£[M_1(u(x,t)) + M_1'(v(x,t))] + £[N_1(v(x,t))]\},\\ \bar{v}(x,s)s^{\alpha} &= s^{\alpha-1}v_0(x) + £[g(x,t)] - \{£[M_2(v(x,t)) + M_2'(u(x,t))] + £[N_2(u(x,t))]\}.\end{aligned}\right\} \tag{23}$$

Using inversion of Laplace transform on equation (23), we get

$$\left.\begin{aligned} u(x,t) &= -\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_1(u(x,t)) + M_1'(v(x,t)) + N_1(v(x,t))]\} + K_1(x,t), \\ v(x,t) &= -\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_2(v(x,t)) + M_2'(u(x,t)) + N_2(u(x,t))]\} + K_2(x,t). \end{aligned}\right\} \tag{24}$$

In the above equation, $K_2(x,t), K_1(x,t)$ are the terms arises due to source terms and initial conditions. Further, we apply the HPM, we find the numerical solution in the power series of ρ:

$$\left.\begin{aligned} u(x,t) &= \sum_{j=0}^{\infty}\rho^j u_j(x,t), \\ v(x,t) &= \sum_{j=0}^{\infty}\rho^j v_j(x,t). \end{aligned}\right\} \tag{25}$$

Also, the nonlinear term occurring in method can be disintegrated in terms of He's polynomials as:

$$\left.\begin{aligned} N_1(v(x,t)) &= \sum_{j=0}^{\infty}\rho^j H_j(\rho), \\ N_2(u(x,t)) &= \sum_{j=0}^{\infty}\rho^j H_j'(\rho). \end{aligned}\right\} \tag{26}$$

In the above equation $H_n(\rho), H_n'(\rho)$ are the He's polynomials as defined in equation (18). Now using equation (18) and (26) in equation (24), we get

$$\left.\begin{aligned} \sum_{j=0}^{\infty}\rho^j u_j(x,t) &= K_1(x,t) - \rho(\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_1(\sum_{j=0}^{\infty}\rho^j u_j(x,t)) + M_1'(\sum_{j=0}^{\infty}\rho^j v_j(x,t)) + \sum_{j=0}^{\infty}\rho^j H_j(\rho)]\}), \\ \sum_{j=0}^{\infty}\rho^j v_j(x,t) &= K_2(x,t) - \rho(\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_2(\sum_{j=0}^{\infty}\rho^j v_j(x,t)) + M_2'(\sum_{j=0}^{\infty}\rho^j u_j(x,t)) + \sum_{j=0}^{\infty}\rho^j H_j'(\rho)]\}). \end{aligned}\right\} \tag{27}$$

Above equation is coupling of Laplace transform technique, HPM and He's polynomials. Now comparing coefficients of identical powers of ρ, we find the approximate solution of concerned nonlinear problem as

$$\begin{aligned} &\rho^0 : u_0(x,t) = K_1(x,t), \\ &\rho^0 : v_0(x,t) = K_2(x,t), \\ &\rho^1 : -\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_1(u_0(x,t)) + M_1'(v_0(x,t)) + H_0(u)]\}, \\ &\rho^1 : -\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_2(v_0(x,t)) + M_2'(u_0(x,t)) + H_0'(u)]\}, \\ &\rho^2 : -\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_1(u_1(x,t)) + M_1'(v_1(x,t)) + H_1(u)]\}, \\ &\rho^2 : -\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_2(v_1(x,t)) + M_2'(u_1(x,t)) + H_1'(u)]\}, \\ &. \\ &. \\ &. \\ &\rho^{j+1} : -\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_1(u_j(x,t)) + M_1'(v_j(x,t)) + H_j(u)]\}, \\ &\rho^{j+1} : -\pounds^{-1}\{\frac{1}{s^{\alpha}}\pounds[M_2(v_j(x,t)) + M_2'(u_j(x,t)) + H_j'(u)]\}. \end{aligned} \tag{28}$$

Using th above approximation and recurrence relation, we can easily find the remaining terms $u_j(x,t), v_j(x,t)$ and hence we find desired solution in form of series. In this way we approximate the exact solution $v(x,t), u(x,t)$ as

$$\left.\begin{aligned} u(x,t) &= \lim_{L\to\infty}\sum_{j=0}^{L} u_j(x,t). \\ v(x,t) &= \lim_{L\to\infty}\sum_{j=0}^{L} v_j(x,t). \end{aligned}\right\} \tag{29}$$

The above series solution converges to the exact solution very fast after few terms.

5. Numerical Examples

In this section of the article, we will apply Laplace transform homotopy method for the fractional order differentiation to solve some fractional order coupled system of PDEs (linear/non-linear) to illustrate the accuracy and applicability of the proposed scheme and compared the obtained results with the exact solutions of the given examples. For the numerical commutations, Mathematica 11.3 software is used.

Example 1. Coupled Burgers' Equation

Let us consider a nonlinear fractional order system of coupled Burgers' equation which have mainly use in describing the nature of sedimentation or evolution of two solute species concentration under the gravitational force in the fluid suspension or colloids phenomenon[37] etc. Coupled Burgers' equation is presented by using some appropriate value of constants in (1), given by

$$\left.\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=\frac{\partial^2 u(x,t)}{\partial x^2}+2\frac{\partial u(x,t)}{\partial x}u(x,t)-\frac{\partial(v(x,t)u(x,t))}{\partial x},\\ \frac{\partial v(x,t)}{\partial t}&=\frac{\partial^2 v(x,t)}{\partial x^2}+2\frac{\partial v(x,t)}{\partial x}v(x,t)-\frac{\partial(v(x,t)u(x,t))}{\partial x},\end{aligned}\right\}\tag{30}$$

with the initial conditions (2) as

$$v(x,0)=\sin(x),\quad u(x,0)=\sin(x).\tag{31}$$

Under the assumption of above conditions the exact solution of above equation is given as $v(x,t)=\exp(-t)\sin(x)$, $u(x,t)=\exp(-t)\sin(x)$. Applying Laplace transform on the above nonlinear PDE together with initial conditions, we have

$$\left.\begin{aligned}£[\frac{\partial u(x,t)}{\partial t}]&=£_t[\frac{\partial^2 u(x,t)}{\partial x^2}+2u(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial(u(x,t)v(x,t))}{\partial x}],\\ £[\frac{\partial v(x,t)}{\partial t}]&=£_t[\frac{\partial^2 v(x,t)}{\partial x^2}+2v(x,t)\frac{\partial v(x,t)}{\partial x}-\frac{\partial(u(x,t)v(x,t))}{\partial x}].\end{aligned}\right\}\tag{32}$$

Now, using equation (8) and (42) in above equation, we get,

$$\left.\begin{aligned}s^{\alpha}\bar{u}(x,s)-\sin(x)s^{\alpha-1}&=£_t[\frac{\partial^2 u(x,t)}{\partial x^2}+2\frac{\partial u(x,t)}{\partial x}u(x,t)-\frac{\partial(v(x,t)u(x,t))}{\partial x}],\\ s^{\alpha}\bar{v}(x,s)-\sin(x)s^{\alpha-1}&=£_t[\frac{\partial^2 v(x,t)}{\partial x^2}+2\frac{\partial v(x,t)}{\partial x}v(x,t)-\frac{\partial(v(x,t)u(x,t))}{\partial x}].\end{aligned}\right\}\tag{33}$$

It can also be written as

$$\left.\begin{aligned}\bar{u}(x,s)&=\frac{\sin(x)}{s}+\frac{1}{s^{\alpha}}£_t[\frac{\partial^2 u(x,t)}{\partial x^2}+2u(x,t)\frac{\partial u(x,t)}{\partial x}-\frac{\partial(u(x,t)v(x,t))}{\partial x}],\\ \bar{v}(x,s)&=\frac{\sin(x)}{s}+\frac{1}{s^{\alpha}}£_t[\frac{\partial^2 v(x,t)}{\partial x^2}+2\frac{\partial v(x,t)}{\partial x}v(x,t)-\frac{\partial(v(x,t)u(x,t))}{\partial x}].\end{aligned}\right\}\tag{34}$$

Applying inverse of Laplace transform and its properties in above equation we get

$$\left.\begin{aligned}u(x,t)&=£_s^{-1}\{\frac{1}{s^{\alpha}}£_t[\frac{\partial^2 u(x,t)}{\partial x^2}+2\frac{\partial u(x,t)}{\partial x}u(x,t)-\frac{\partial(v(x,t)u(x,t))}{\partial x}]\}+\sin(x),\\ v(x,t)&=£_s^{-1}\{\frac{1}{s^{\alpha}}£_t[\frac{\partial^2 v(x,t)}{\partial x^2}+2\frac{\partial v(x,t)}{\partial x}v(x,t)-\frac{\partial(v(x,t)u(x,t))}{\partial x}]+\sin(x).\end{aligned}\right\}\tag{35}$$

Further, using HPM and He's polynomial for nonlinear term as

$$\left.\begin{aligned}u(x,t)&=\sum_{j=0}^{\infty}\rho^j u_j(x,t),\\ v(x,t)&=\sum_{j=0}^{\infty}\rho^j v_j(x,t),\end{aligned}\right\}\tag{36}$$

and for nonlinear term, we have

$$\left.\begin{aligned}\sum_{j=0}^{\infty}\rho^j H_j(\rho)&=N_1 v(x,t),\\ \sum_{j=0}^{\infty}\rho^j H_j'(\rho)&=N_2 u(x,t).\end{aligned}\right\}\tag{37}$$

In general He's polynomial is given by equation (18) and few He's polynomials for the coupled Burgers' equation for $\alpha=1$ can be found as

$$\begin{aligned}&H_0(u)=(u_0^2)_x-v_0.(u_0)_x-(v_0)_x.u_0,\\ &H_0'(u)=(v_0^2)_x-u_0.(v_0)_x-v_0.(u_0)_x,\\ &H_1(u)=(2u_0u_1)_x-v_1.(u_0)_x-v_0.(u_1)_x-(v_0)_x.u_1-u_0.(v_1)_x,\\ &H_1'(u)=(2v_0v_1)_x-(v_0)_x.u_1-(v_1)_x.u_0-v_1.(u_0)_x-v_0.(u_1)_x,\\ &.\\ &.\\ &.\end{aligned}\tag{38}$$

Susbstituting equations (36)-(38) into (35) and comparing like power of ρ we have the approximation and recurrence relation as

$$
\begin{aligned}
&\rho^0 : u_0(x,t) = \sin(x),\\
&\rho^0 : v_0(x,t) = \sin(x)\\
&\rho^1 : u_1(x,t) = \pounds^{-1}\{\frac{1}{s}\pounds[(u_0)_{xx} + H_0(u)]\} = -\sin(x)t,\\
&\rho^1 : v_1(x,t) = \pounds^{-1}\{\frac{1}{s}\pounds[(v_0)_{xx} + H_0'(u)]\} = -\sin(x)t,\\
&\rho^2 : u_2(x,t) = \pounds^{-1}\{\frac{1}{s}\pounds[(u_1)_{xx} + H_1(u)]\} = \frac{1}{2}t^2\sin(x),\\
&\rho^2 : v_2(x,t) = \pounds^{-1}\{\frac{1}{s}\pounds[(v_1)_{xx} + H_1'(u)]\} = \frac{1}{2}t^2\sin(x),\\
&.\\
&.\\
&.
\end{aligned}
\tag{39}
$$

Thus from equation (36) we get the series solution as

$$
\left.
\begin{aligned}
u(x,t) &= \sin(x) - \sin(x)t + \frac{1}{2}\sin(x)t^2 + \cdots,\\
v(x,t) &= \sin(x) - \sin(x)t + \frac{1}{2}\sin(x)t^2 + \cdots.
\end{aligned}
\right\}
\tag{40}
$$

The approximate series solution in (40) rapidly converges to exact solution after few approximation.
Absolute errors between the approximate solution and exact solution is given in Figures 1-2 for the different values of x and t. These figures clearly confirm that the approximate solution of $v(x,t)$ and $u(x,t)$ are very close to the exact solution, which ensure the accuracy of numerical technique and ensure the method is accurate and effective to find the solution of NPDEs.

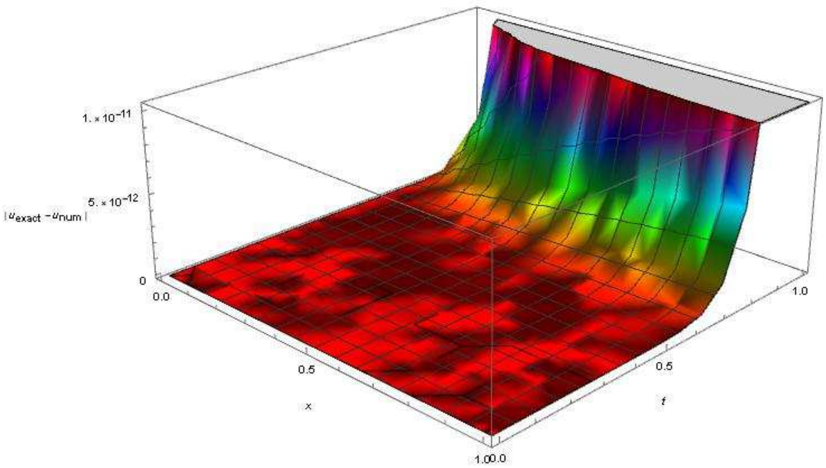

Fig. 1. Error graph between the numerical and exact solutions of $u(x,t)$ vs. t and x.

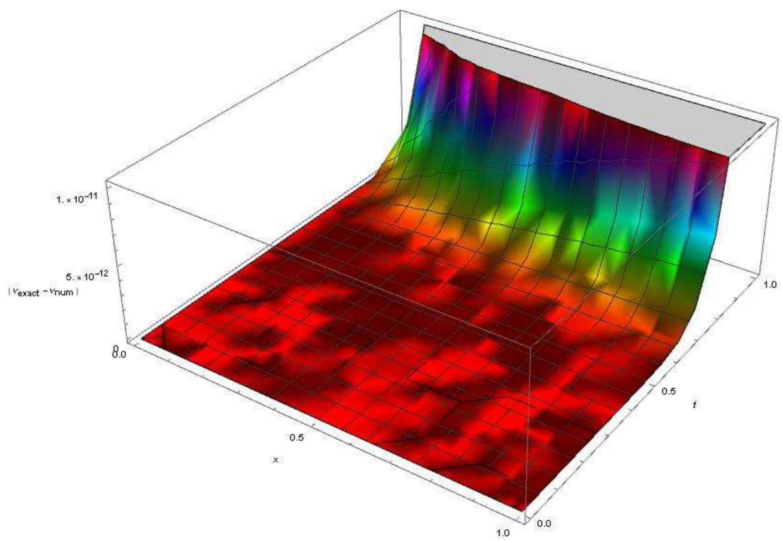

Fig. 2. Error graph between the numerical and exact solutions of $v(x,t)$ vs. t and x.

Table 1. Comparison of L_∞ and L_2 errors for $u(x,t)$

t	Proposed Method		Pandey et al.[19]		Mittal et al.[38]	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.01	1.04×10^{-16}	1.11×10^{-16}	5.62×10^{-8}	1.13×10^{-7}	3.49×10^{-4}	2.90×10^{-4}
0.1	1.09×10^{-16}	1.11×10^{-16}	1.38×10^{-7}	1.89×10^{-7}	3.78×10^{-4}	2.65×10^{-4}
0.5	2.43×10^{-15}	1.59×10^{-14}	6.62×10^{-8}	1.19×10^{-7}	4.47×10^{-4}	1.07×10^{-4}
1	4.38×10^{-13}	1.26×10^{-12}	4.22×10^{-8}	7.07×10^{-8}	7.87×10^{-4}	3.76×10^{-4}

Table 2. Comparison of L_∞ and L_2 errors for $v(x,t)$

t	Proposed Method		Pandey et al.[19]		Mittal et al.[38]	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.01	1.04×10^{-16}	1.10×10^{-16}	5.62×10^{-8}	1.13×10^{-7}	3.49×10^{-4}	2.90×10^{-4}
0.1	1.06×10^{-16}	1.11×10^{-16}	1.38×10^{-7}	1.89×10^{-7}	3.78×10^{-4}	2.65×10^{-4}
0.5	2.84×10^{-15}	1.59×10^{-14}	6.62×10^{-8}	1.19×10^{-7}	4.47×10^{-4}	1.07×10^{-4}
1	4.84×10^{-13}	1.27×10^{-12}	4.22×10^{-8}	7.07×10^{-8}	7.87×10^{-4}	3.76×10^{-4}

Table 3. Comparison of L_∞ and L_2 errors for $u(x,t)$

t	Proposed Method		Pandey et al.[19]		Oruc et al.[39]	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.5	2.76×10^{-14}	2.91×10^{-14}	2.53×10^{-6}	3.88×10^{-5}	6.79×10^{-4}	4.15×10^{-5}
1	3.9×10^{-15}	2.40×10^{-14}	2.59×10^{-5}	3.96×10^{-5}	1.33×10^{-3}	8.21×10^{-5}

The L_2 and L_∞ errors of example 1 for different t for $v(x,t)$ and $u(x,t)$ for our proposed numerical scheme and the method proposed in [19, 38] are given in the Table 1 and 2. And on the observation of these tables we can strongly says that proposed numerical technique is better than method proposed in [19, 38].

Example 2. Another form of Coupled Burgers' Equation

Let us consider a another form of nonlinear fractional order system of coupled Burgers' equation. This form of Coupled Burgers' equation is also presented by using some appropriate value of constants in (1), given by

$$\left.\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=\frac{\partial^2 u(x,t)}{\partial x^2}+2\frac{\partial u(x,t)}{\partial x}u(x,t)-0.1\frac{\partial(v(x,t)u(x,t))}{\partial x},\\ \frac{\partial v(x,t)}{\partial t}&=\frac{\partial^2 v(x,t)}{\partial x^2}+2\frac{\partial v(x,t)}{\partial x}v(x,t)-0.3\frac{\partial(v(x,t)u(x,t))}{\partial x},\end{aligned}\right\} \tag{41}$$

with the initial conditions (2) as

$$v(x,0)=0.05(0.5-\tanh(0.0275(20(x-0.5)))),\quad u(x,0)=0.05(1-\tanh(0.0275(20(x-0.5)))) \tag{42}$$

Under the assumption of above conditions the exact solution of above equation is given as:
$v(x,t)=0.05(0.5-\tanh(0.0275(20(x-0.5)-0.055t)))$, $u(x,t)=0.05(1-\tanh(0.0275(20(x-0.5)-0.055t)))$. Using the proposed scheme in the above system we find the approximate numerical solution of example 2. Absolute error between the approximate solution and exact solution is given in Figures 3-4 for the different values of x and t. These figures clearly confirm that the approximate solution of $v(x,t)$ and $u(x,t)$ are very close to the exact solution, which ensure the efficiency of our proposed method and ensure the method is accurate and effective.
The analysis of errors for example 2 for various values of x,t for $v(x,t)$ and $u(x,t)$ for our proposed numerical scheme and the method proposed in [19] are given in the Table 3 and 4. And on the observation of these tables we can strongly says that numerical method is efficient than method proposed in [19].

Example 3.

Let us consider a system of coupled linear homogeneous PDE which can be found by giving some particular values to the constants in our concerned coupled system (1) as

$$\left.\begin{aligned}\frac{\partial^{\alpha_1}u(x,t)}{\partial t^{\alpha_1}}+\frac{\partial^{\beta_1}v(x,t)}{\partial x^{\beta_1}}&=0,\\ \frac{\partial^{\alpha_2}v(x,t)}{\partial t^{\alpha_2}}+\frac{\partial^{\beta_2}u(x,t)}{\partial x^{\beta_2}}&=0,\end{aligned}\right\} \tag{43}$$

with the given initial condition $v(x,0)=e^{-x}, u(x,0)=e^{x}$. For the value $\alpha_1=\alpha_2=\beta_1=\beta_2=1$, the exact solution for above homogeneous coupled system is [40]

$$u(x,t)=\cosh(t)e^{x}+\sinh(t)e^{-x},\quad v(x,t)=\cosh(t)e^{-x}-\sinh(t)e^{x}. \tag{44}$$

Using the proposed scheme in the above system we find the approximate numerical solution of example 3. Absolute error between the approximate solution and exact solution is given in Figures 5-6 for the different values of x and t. These figures clearly confirm that the approximate solution of $v(x,t)$ and $u(x,t)$ are very close to the exact solution, which ensure the efficiency of our proposed method and ensure the method is accurate and effective.

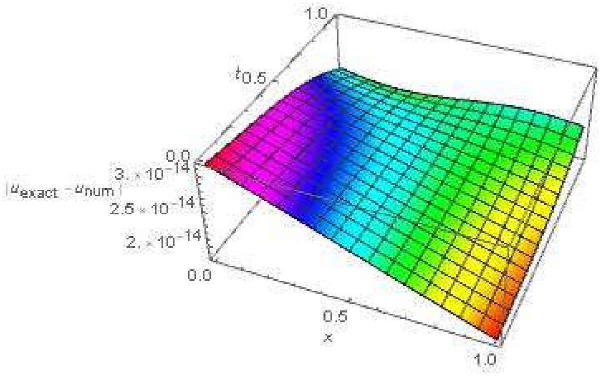

Fig. 3. Error graph between the numerical and exact solutions of $u(x,t)$ vs. t and x.

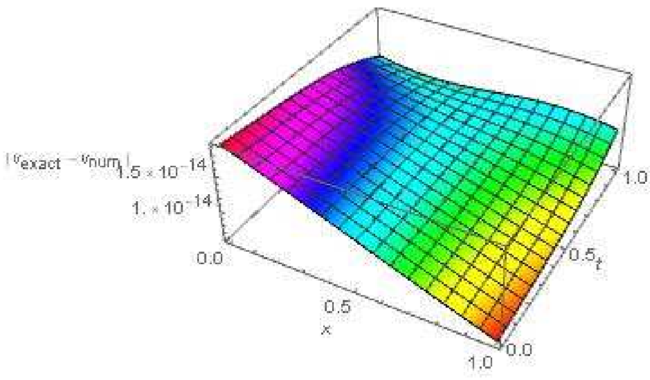

Fig. 4. Error graph between the numerical and exact solutions of $v(x,t)$ vs. t and x.

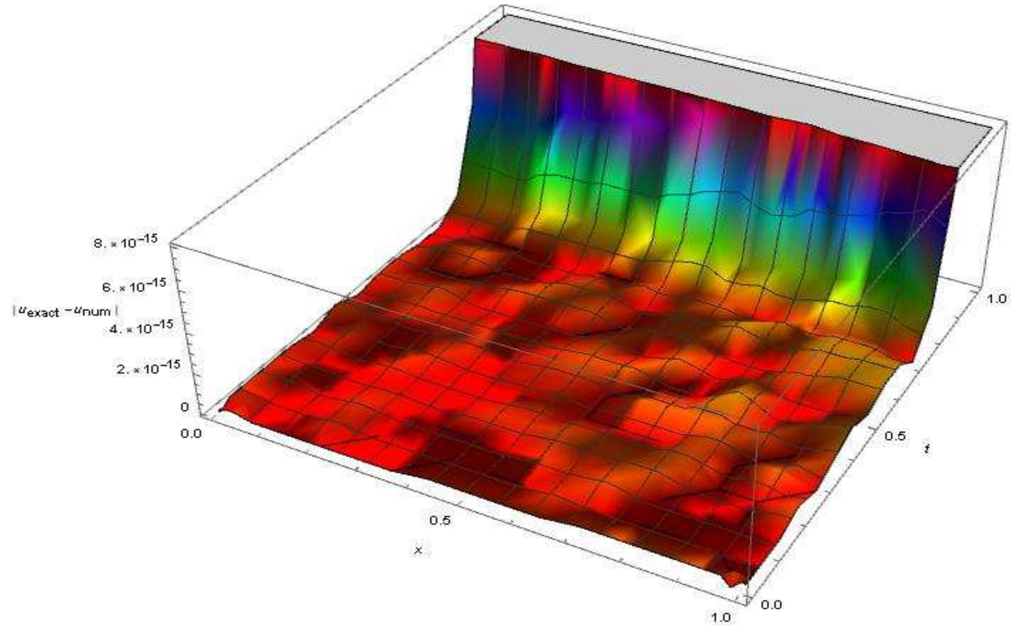

Fig. 5. Error graph between the numerical and exact solutions of $u(x,t)$ vs. t and x.

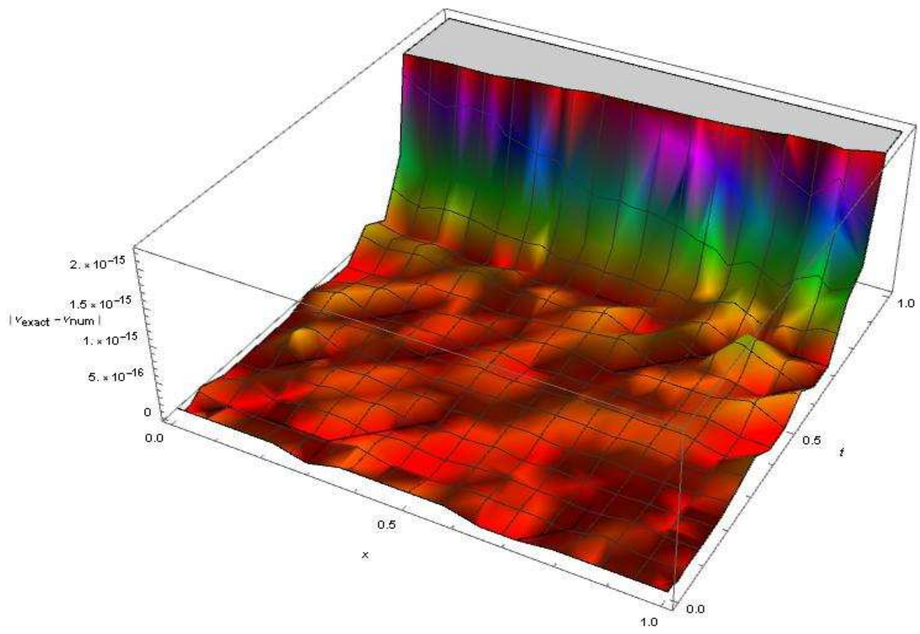

Fig. 6. Error graph between the numerical and exact solutions of $v(x,t)$ vs. t and x.

Table 4. Comparison of L_∞ and L_2 errors for $v(x,t)$

t	Proposed Method		Pandey et al.[19]		Oruc et al.[39]	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.5	1.1×10^{-14}	1.73×10^{-14}	1.21×10^{-6}	2.04×10^{-6}	2.03×10^{-4}	9.39×10^{-6}
1	1.1×10^{-14}	1.27×10^{-14}	1.25×10^{-6}	2.10×10^{-5}	9.85×10^{-4}	4.12×10^{-5}

Table 5. Comparison between the absolute errors for $u(x,t)$ for different x at $t=0.1$

x	Proposed Method	Method given in [40]
	$\lvert u_{exact}(x,t)-u_{num}(x,t)\rvert$	$\lvert u_{exact}(x,t)-u_{num}(x,t)\rvert$
0.1	2.22×10^{-16}	1.9×10^{-10}
0.3	2.21×10^{-16}	2.6×10^{-8}
0.5	3.28×10^{-16}	8.4×10^{-7}
0.7	4.44×10^{-16}	3.8×10^{-7}
0.9	3.86×10^{-16}	5.4×10^{-6}

Table 6. Comparison between the absolute errors for $v(x,t)$ for different x at $t=0.1$

x	Proposed Method	Method given in[40]
	$\lvert u_{exact}(x,t)-u_{num}(x,t)\rvert$	$\lvert u_{exact}(x,t)-u_{num}(x,t)\rvert$
0.1	1.11×10^{-16}	1.0×10^{-9}
0.3	1.11×10^{-16}	2.0×10^{-8}
0.5	1.28×10^{-16}	7.8×10^{-7}
0.7	1.09×10^{-16}	6.5×10^{-7}
0.9	5.55×10^{-17}	2.9×10^{-6}

Table 7. Comparison between the absolute errors for $u(x,t)$ and $v(x,t)$ for different x at $t=1$

x	Proposed Method	Proposed Method
	$\lvert u_{exact}(x,t)-u_{num}(x,t)\rvert$	$\lvert v_{exact}(x,t)-v_{num}(x,t)\rvert$
0.1	5.59×10^{-15}	4.01×10^{-15}
0.3	6.07×10^{-15}	3.18×10^{-15}
0.5	8.08×10^{-15}	2.46×10^{-15}
0.7	9.18×10^{-15}	1.82×10^{-15}
0.9	1.19×10^{-14}	1.28×10^{-15}

The analysis of errors for example 3 for various values of x,t for $v(x,t)$ and $u(x,t)$ for our proposed numerical scheme and the method proposed in [40] are given in the Table 5, 6 and 7. And on the observation of these tables we can strongly says that numerical method is more efficient than method proposed in [40].

6. Results and Simulations for Proposed Model

In the previous section the accuracy and efficiency of the proposed scheme is justified. The excellent efficiency of this method motivated the authors to solve the system of coupled time fractional order advection-diffusion equation. The diffusive nature of approximate numerical solutions $v(x,t)$ and $u(x,t)$ vs the soil column length x at $t=0.5$ are given in the Figures 7-10 for various value of fractional parameter in the non-conservative systems.

In Figures 7-10, the variation in solute concentration for various particular cases of time fractional parameters are given. The solution profile decreases as the coupled system approaches to fractional order from integer order. Damping nature of solute profiles are easily justified for the source terms ($k_1=k_2=1$) in comparison of sink terms ($k_1=k_2=-1$).

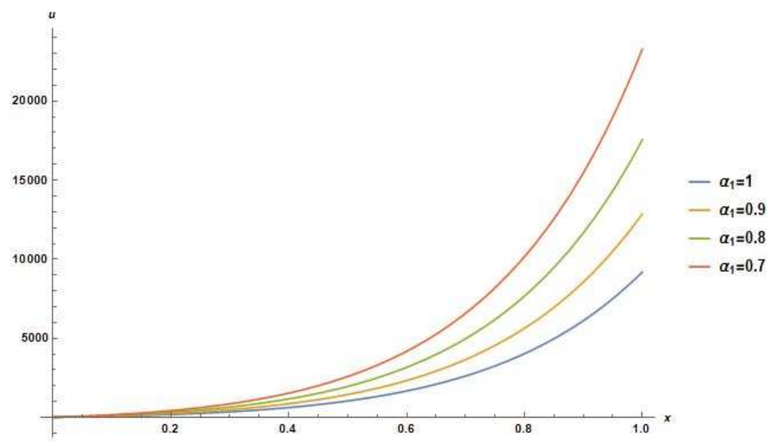

Fig. 7. Graph of the solute species $u(x,t)$ vs. x for different α_1 and $k_1=k_2=1,\alpha_2=1$ at $t=0.5$.

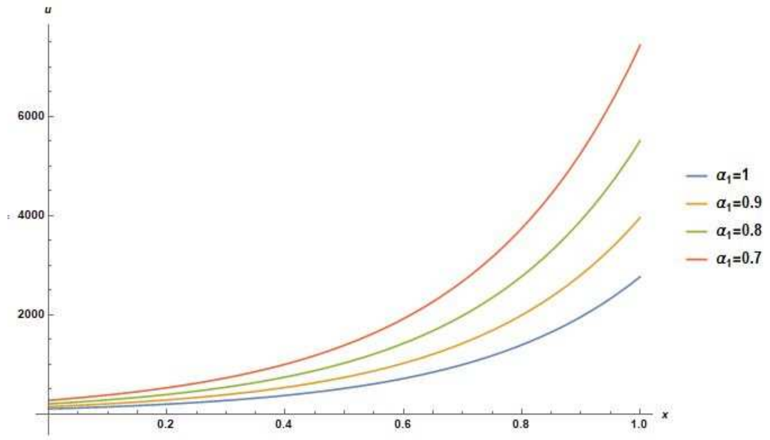

Fig. 8. Graph of the solute species $u(x,t)$ vs. x for different α_1 and $k_1 = k_2 = -1, \alpha_2 = 1$ at $t = 0.5$.

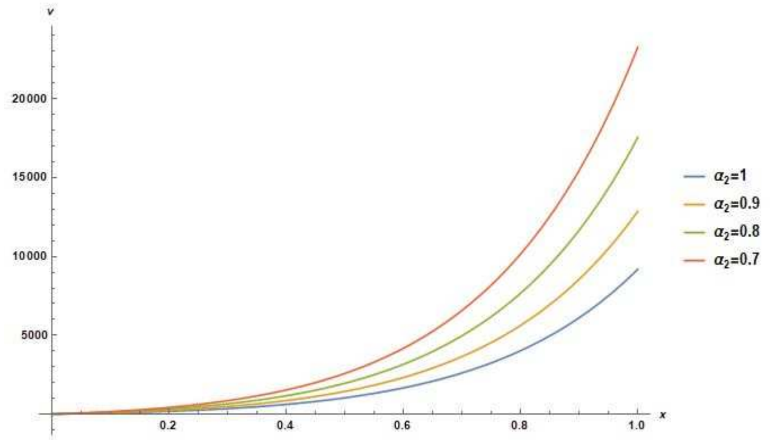

Fig. 9. Graph of the solute species $v(x,t)$ vs. x for different α_2 and $k_1 = k_2 = 1, \alpha_1 = 1$ at $t = 0.5$.

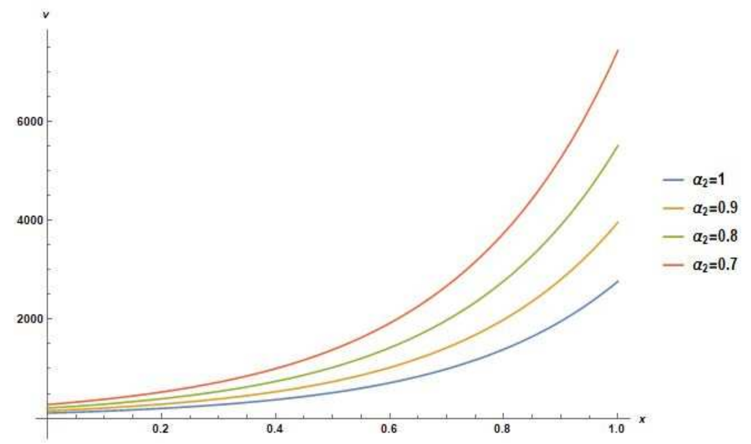

Fig. 10. Graph of the solute species $v(x,t)$ vs. x for different α_2 and $k_1 = k_2 = -1, \alpha_1 = 1$ at $t = 0.5$.

7. Conclusion

In this article, an efficient numerical technique has been proposed by using homotopy perturbation method and He's polynomials for the approximate numerical solution of nonlinear multiterm time fractional order advection-reaction-diffusion equation. The proposed method have been justified on some particular cases of concerned coupled system which shows the high efficiency and accuracy of numerical scheme. The present scientific contribution has achieved three important consequences, which is the demonstration of the convergence of given numerical technique for the solute species $v(x,t)$ and $u(x,t)$ for different fractional time derivatives, the graphical presentation of effect of reaction term on solute profiles with respect to the soil column length at $t = 0.5$ and last one is demonstration of damping behavior of solute profile in effect of reaction term as the coupled system goes to fractional order from integer order. The future scope of the author is to find the numerical solution of wide range of mathematical models of physical phenomenas viz., 2D-3D problems.

Author Contributions

P. Pandey, S. Kumar and J.F. Gómez-Aguilar developed the mathematical modeling and examined the theory validation. The numerical method was worked out by P. Pandey, S. Kumar. P. Pandey and J.F. Gómez-Aguilar analyzed the data and numerical simulations; S. Kumar and J.F. Gómez-Aguilar polished the language and were in charge of technical checking. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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