J. Appl. Comput. Mech., 5(3) (2019) 552-562
DOI: 10.22055/JACM.2018.27589.1418

ISSN: 2383-4536
jacm.scu.ac.ir

M&MoCS

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

Cracking Elements Method for Simulating Complex Crack Growth

Zizheng Sun[1], Xiaoying Zhuang[2,3,4], Yiming Zhang[1]

[1] School of Civil and Transportation Engineering, Hebei University of Technology
Xiping Road 5340, 300401 Tianjin, China, Emails: Zizheng.Sun@hebut.edu.cn, Yiming.Zhang@hebut.edu.cn

[2] Institute of Continuum Mechanics, Leibniz Universit¨at Hannover, Appelstraße 11, 30157 Hannover, Germany, Xiaoying.Zhuang@gmail.com

[3] Department of Geotechnical Engineering, Tongji University, Siping Road 1239, 200092 Shanghai, China

[4] State Key Laboratory for Disaster Reduction in Civil Engineering, Tongji University, Siping Road 1239, 200092 Shanghai, China

Received November 12 2018; Revised November 21 2018; Accepted for publication December 09 2018.
Corresponding author: Yiming Zhang, Yiming.Zhang@hebut.edu.cn

Abstract. The cracking elements method (CEM) is a novel numerical approach for simulating fracture of quasi-brittle materials. This method is built in the framework of conventional finite element method (FEM) based on standard Galerkin approximation, which models the cracks with disconnected cracking segments. The orientation of propagating cracks is determined by local criteria and no explicit or implicit representations of the cracks' topology are needed. *CEM does not need remeshing technique, cover algorithm, nodal enrichment or specific crack tracking strategies*. The crack opening is condensed in local element, greatly reducing the coding efforts and simplifying the numerical procedure. This paper presents numerical simulations with CEM regarding several benchmark tests, the results of which further indicate the capability of CEM in capturing complex crack growths referring *propagations of existed cracks as well as initiations of new cracks*.

Keywords: Cracking elements method, Fracture analysis, Quasi-brittle material, Complex crack growth.

1. Introduction

Great efforts are paid by researchers for numerically capturing discontinuity of solids, because of the hazardousness of discontinuity in engineering applications such as cracks and shear bands as well as the challengingness of mathematical modelling of it. For quasi-brittle materials, during damaging processes, emergence of macro cracks is accompanied by the unloading of surrounding undamaged regions and the width of the fracture process zone will reduce from finite width into zero [1,2]. This fact indicates the fractures of quasi-brittle materials are highly localized and anisotropic [3-8], showing so called strong discontinuity behavior.

In last decades, with the understanding of fracture processes and developing of computational mechanics, many numerical methods have been presented for numerically reproducing and simulating the fracture process, including remeshing and interface element method [9-13], multi-field mixed-mode formulation [14-16], gradient based and phase field models [17-29], numerical manifold method [30-35], Strong Discontinuity embedded Approach (SDA) [36-47], eXtended Finite Element Method (XFEM) and phantom node method [48-56], meshfree methods [57-67], cracking particles method [68-71] and peridynamics based methods [72-83], to mention a few.

All of these methods have their own advantages and disadvantages. As observed by us, the methods focusing on precise describing the crack tips such as remeshing and XFEM assume continuous crack path, which need crack tracking strategies [84-89]. Despite of the powerfulness of tracking strategies for eliminating stress locking [90], tracking strategies could be inflexible and inefficient for simulating complex crack growth. The gradient based and phase field models are good at simulating the initiation and development of damage regions, which however cannot capture the anisotropic damage properties

Published online: April 05 2019

such as orientation and opening of the crack. The multi-field mixed-mode formulation and meshless methods are generally flexible and powerful at simulating complex crack, but they are generally inefficient comparing to classical FEM approaches. Peridynamics based methods use control equations formulated in an integral form but not partial differential form, to avoid dealing with singularity problems in fractures analysis. Nevertheless, the material parameters such as "strength" and "fracture energy" must be firstly transformed then used in peridynamics [79].

Cracking elements method (CEM) [91] is a novel numerical approach for fracture analysis based on the Statically Optimal Symmetric formulation of the SDA (SDA-SOS) [92-94]. As been pointed out in [95, 96], SDA-SOS uses the equivalent crack length, naturally supporting self-propagating cracks. The work presented in [97] solved the stress-locking problem of SDA-SOS by using elements with nonlinear interpolations for the displacement fields, paving the way for CEM's formulation. Like the other types of SDA, CEM is built in the framework of the finite element method (FEM). When modifying conventional FEM code into CEM, no remeshing technique, cover algorithm, nodal enrichment or specific crack tracking strategy are needed and the standard Galerkin method can be applied, greatly reducing the coding efforts. On the other hand, unlike the other types of SDA, CEM uses disconnected cracking segments for representing crack paths. There are no differences between elements with crack tips and elements on crack paths, then no explicit or implicit descriptions of crack's topology are needed. Hence, cracks in CEM can grow freely and naturally based on the present stress states. Local criteria are proposed for determining the initiation and orientation of the cracks, making this method very flexible for simulating complex crack growth.

In this work, we further investigate CEM in simulating complex crack growth, regarding propagations of existed cracks as well as initiations of new cracks. The remaining parts of this paper are organized as follows: In Section 2, the formulation of CEM are briefly introduced. Numerical examples are presented in Section 3 including Brazilian disk splitting test, plate with one hole and bending test with three holes. Irregular discretizations are always used for demonstrating the robustness of CEM. The paper closes with concluding remarks given in Section 4.

2. The formulation

2.1 Constitutive relation

In this paper, mixed-mode traction-separation [98-103] law is used for describing the relationship of the crack opening and the traction between two surfaces of the crack as

$$\zeta_{\text{eq}} = \sqrt{\zeta_n^2 + \zeta_t^2} \tag{1}$$

where ζ_n and ζ_t are the crack opening or crack width along normal and parallel directions of the crack path, with corresponding unit vector denoted as **n** and **t** and the traction components along **n** and **t**, T_n and T_t are obtained as

$$T_n = T_{eq}\frac{\zeta_n}{\zeta_{\text{eq}}}, T_t = T_{eq}\frac{\zeta_t}{\zeta_{\text{eq}}} \tag{2a}$$

with

$$T_{eq}\left(\zeta_{\text{eq}}\right) = \begin{cases} TL\left(\zeta_{\text{eq}}\right) = f_t \exp\left(-\frac{f_t}{G_f}\zeta_{\text{eq}}\right) & \text{loading} \\ TU\left(\zeta_{\text{eq}}\right) = \frac{T_{mx}}{\zeta_{mx}}\zeta_{\text{eq}} & \text{unloading/reloading} \end{cases} \tag{2b}$$

where f_t is the uniaxial tensile strength and G_f is the fracture energy, ζ_{mx} is the maximum crack opening the crack ever experienced, and $T_{mx} = TL\left(\zeta_{mx}\right)$ is the corresponding traction see [93, 102].

Correspondingly, the relationship between the tractions differentials dT_n, dT_t and the crack opening differentials ζ_n and ζ_t is described by

$$\begin{bmatrix} dT_n \\ dT_t \end{bmatrix} = \mathbf{D}\begin{bmatrix} d\zeta_n \\ d\zeta_t \end{bmatrix} \tag{3}$$

with

$$\mathbf{D} = -\frac{T_{eq}}{\zeta_{\text{eq}}}\begin{bmatrix} \frac{\zeta_n^2}{\zeta_{eq}^2} + \frac{f_t\zeta_n^2}{G_f\zeta_{\text{eq}}} - 1 & \frac{\zeta_n\zeta_t}{\zeta_{eq}^2} + \frac{f_t\zeta_n\zeta_t}{G_f\zeta_{\text{eq}}} \\ \frac{\zeta_n\zeta_t}{\zeta_{eq}^2} + \frac{f_t\zeta_n\zeta_t}{G_f\zeta_{\text{eq}}} & \frac{\zeta_t^2}{\zeta_{eq}^2} + \frac{f_t\zeta_t^2}{G_f\zeta_{\text{eq}}} - 1 \end{bmatrix} \text{ for loading,} \tag{4a}$$

and

$$\mathbf{D} = \frac{T_{mx}}{\zeta_{\text{mx}}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for unloading/reloading,} \tag{4b}$$

in which obviously **D** keeps symmetric.

2.2 Cracking elements formulation

Cracking elements method is built based on Statically Optimal Symmetric formulation of the SDA (SDA-SOS). Similar to other types of SDA and XFEM, the basic idea of CEM is to decompose the displacement field with regular part and the other part dedicated by the discontinuity, which is smoothed by a differentiable function $\varphi(\mathrm{x})$, see [102]. Then the strain field is corresponding decomposed by the total strain contributed by the original displacements and the enhanced strain dedicated by the displacement jumps (crack opening) ζ_n and ζ_t. In the FEM formulation, 8 node quadrilateral element is considered for CEM for avoiding stress locking and mesh bias [91, 97]. Hereby, $\nabla\varphi^{(e)}$ is introduced for bridging the displacement jumps to the strain field. $\nabla\varphi^{(e)}$ is a vector parallel to $\mathrm{n}^{(e)}$ the length of which depends only on the shape of the element. From the classical point of view, the length of $\nabla\varphi^{(e)}$ equals to the reciprocal of characteristic length [104]. As mentioned, our approach belongs to the SDA-SOS, which is very distinct from XFEM and the other types of SDA i.e., Kinematically Optimal Symmetric (SDA-KOS) formulation [41,106] and Statically and Kinematically Optimal Nonsymmetric (SDA-SKON) formulation [107-114], which firstly determine φ_i then calculate $\nabla\varphi^{(e)} = \nabla N_i^{(e)} \cdot \varphi_i$. In SDA-SOS, it is unnecessary to know the shape of function φ. Considering the equivalence of force between discrete and embedded models [91,115-117], the local crack opening is condensed and calculated by solving local balance equation. After solving, the tangent moduli [118,119] is also automatically determined. More details can also be found in [91, 97, 102].

2.3 Local criteria for crack propagation

In CEM, continuity of crack path is abandoned, bringing great flexibility. In local element, the orientation of crack $\mathrm{n}^{(e)}$ keeps parallel to the maximum total strain at the center point. Hence, the orientation of the crack only depends on the regular displacement field but not the displacement jumps.

During calculation, all the non-cracked elements in the whole computation domain are separated into two subdomains as i) the propagation domain and ii) the potential root domain, the definitions of which only depend on the neighboring relationship of the cracked and non-cracked elements. The propagating domain is composed by the elements sharing at least one edge with the cracked elements and all the other non-cracked elements belong to the potential root domain. With the propagation of the cracks (or cracked elements), the propagating domain will expand, see Figure 1 [91, 102].

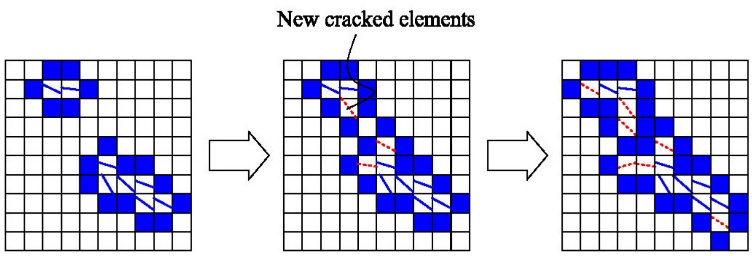

Fig. 1. The expansion of the propagation domain

2.4 FEM implementation and Newton iteration

Considering the static loading condition and the standard displacement-based FEM [120], following equilibrium equation is obtained

$$\mathrm{KU} = \mathrm{F} \tag{5}$$

where U is the vectors of displacement at the nodes. F is the vector of externally applied loads. K is the stiffness matrix as

$$\mathrm{K} = \sum_e \int \left(B^{(e)}\right)^T \mathrm{K}^e \mathrm{B}^{(e)} d\,(e) \tag{6}$$

in which $\int d\,(e)$ denotes the integral of the corresponding functions in element *e*. In Equation 6,

$$\mathrm{B}^{(e)} = \left[\mathrm{B}_1^{(e)} \cdots \mathrm{B}_8^{(e)}\right], \text{ where } \mathrm{B}_i^{(e)} = \begin{bmatrix} \dfrac{\partial N_i^{(e)}}{\partial x} & \\ & \dfrac{\partial N_i^{(e)}}{\partial y} \\ \dfrac{\partial N_i^{(e)}}{\partial y} & \dfrac{\partial N_i^{(e)}}{\partial x} \end{bmatrix} \tag{7}$$

and K^e is the tangent moduli written in matrix form with Voigt notation. Regarding the Newton-Raphson iteration at load step *j*, iteration step *l*, following equation is introduced

$$U_{j,l} = U_{j-1} + \Delta U_{j,l-1} + \Delta\Delta U_l \tag{8}$$

where $\Delta\Delta U_l$ is the unknown, and all the parameter with subscript "$j-1$" and "$j,l-1$" are known. Then considering the local balance relationship, following linearized equation is obtained

$$K_S \cdot (\Delta\Delta U_l) = R_S = F - R_{j,l-1} \tag{9a}$$

where

$$R_{j,l-1} = \sum_e \int \left(B^{(e)}\right)^T S_{j,l-1} d(e) \tag{9b}$$

in which $S_{j,l-1}$ is σ written in vector form for cracking elements at Newton iteration step $l-1$ and time step j. After solving Equation 9, $\Delta U_{j,l}$ is updated as $\Delta U_{j,l} = \Delta U_{j,l-1} + \Delta\Delta U_l$. This iteration procedure is illustrated in ref. [102]. Since the displacement jump is condensed out at the material level, it is very easy to code and implement the approach into conventional FEM code. About the iteration procedures, the interested reader is also referred to refs. [121-130].

3. Numerical simulations

Plane-stress assumptions hold for all the numerical examples presented in this section.

3.1 Brazilian disk Splitting test

Brazilian disk splitting test [131] is commonly used to determine the tensile strength of quasi-brittle materials such as rock and concrete. The simulated model is shown in Figure 2 with incremental displacement prescribed. The relationship between the ultimate loading per unit thickness $F_{\max}$ and tensile strength f_t is determined by

$$F_{\max} = \frac{\pi D f_t}{2} \tag{10}$$

where D is the diameter of the specimen. Hence, for this example the theoretical maximum load is $F_{\max}$ = 471.24kN per unit thickness. The force-displacement curves and the results of $F_{\max}$ are shown in Figure 3, proving that CEM provides very agreeable results with ignorable mesh bias. Furthermore, the final crack paths (cracked elements) regarding different discretizations are shown in Figure 4, indicating that CEM gives disconnected but reasonable crack paths.

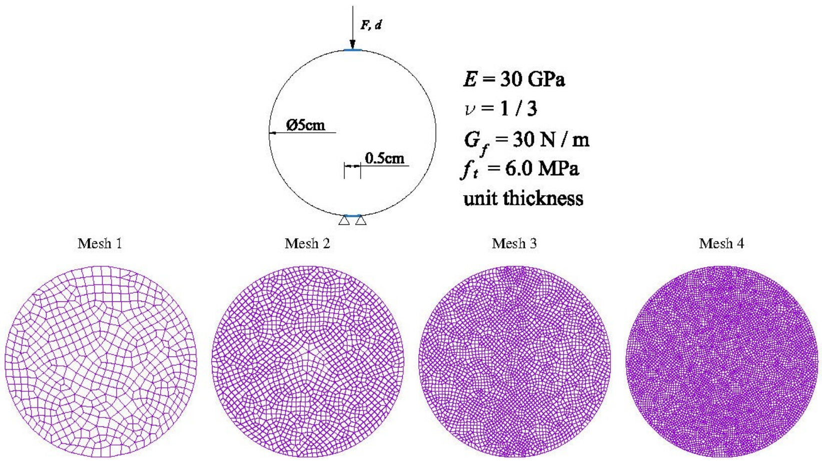

Fig. 2. Brazilian disk splitting test: model and discretizations

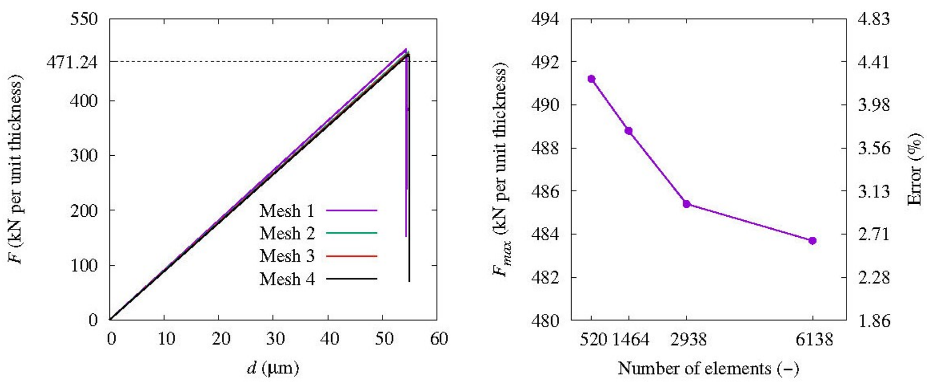

Fig. 3. Brazilian disk splitting test: force-displacement curves and the results of F_{max}

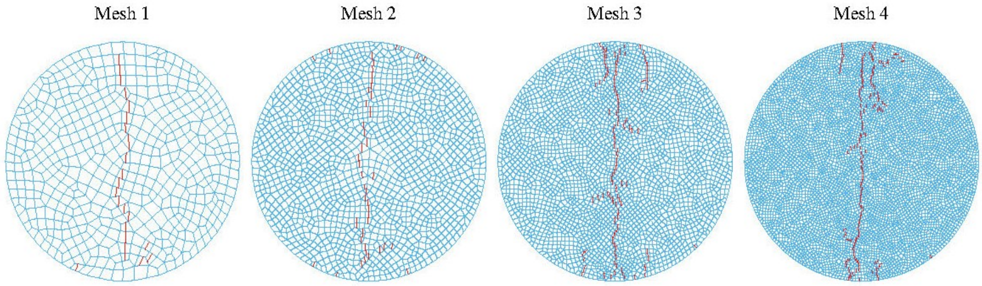

Fig. 4. Brazilian disk splitting test: cracked elements regarding different discretizations (cracked elements)

3.2 Plate with one hole

The second example is a plate with one hole made from brittle material, see Figure 5 for the model. There are three holes in the plate and the two holes with diameter 1 cm are used for fixing supports and transferring constant incremental displacement. The force-displacement curve is shown in Figure 6, comparing with the results published in [132,133]. The cracked elements are shown in Figure 7, which is agreeable with the experimental result shown in [133].

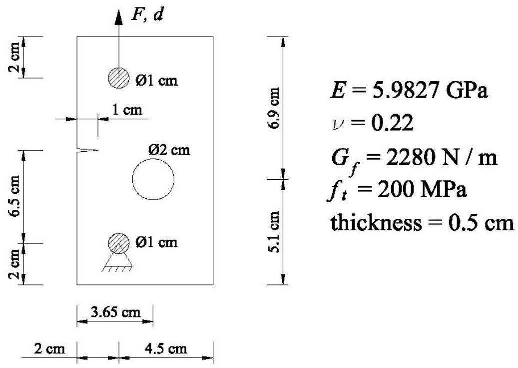

Fig. 5. Plate with one hole: model

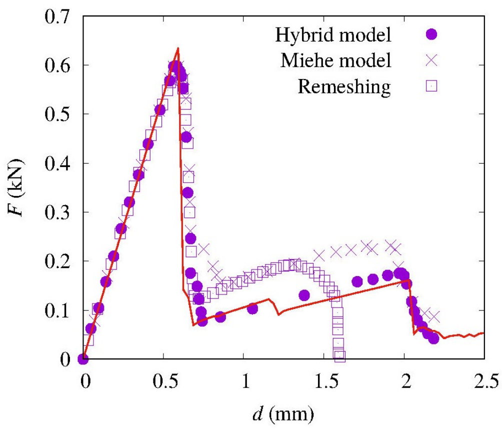

Fig. 6. Plate with one hole: force-displacement curve, comparing with the results published in [132, 133]

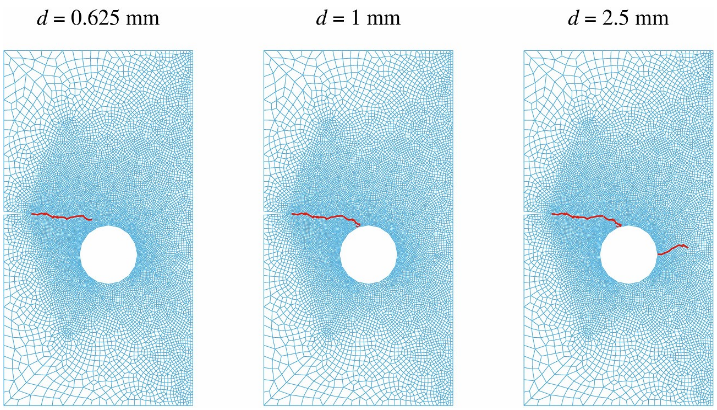

Fig. 7. Plate with one hole: cracked elements at different steps

3.3 Bending beam with three holes

The third example is a bending beam with three holes made from polymethylmethacrylate (PMMA), as a typical brittle material, which was investigated in [17,134-136]. The model is shown in Figure 8. Two cases are simulated regarding the position of the notch as: i) a=15.24 cm, b=2.54cm; and ii) a=12.7cm, b=3.81cm. The arc-length method [137] is used for inducing a constant increment for the crack mouth opening displacement (CMOD) of the notch as 5 μm. The force-displacement curves are shown in Figure 9. The crack paths are shown in Figures 10 and 11, indicating that CEM can capture the different crack patterns in this case. On the other hand, oscillations of force-displacement curves are found, especially in case II, which is mainly caused by the CMOD controlled arc-length method. An energy-based arc-length method [138,139] is under investigation.

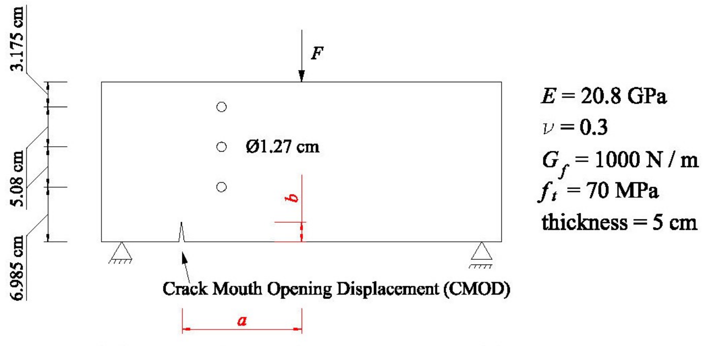

Fig. 8. Bending beam with three holes: model

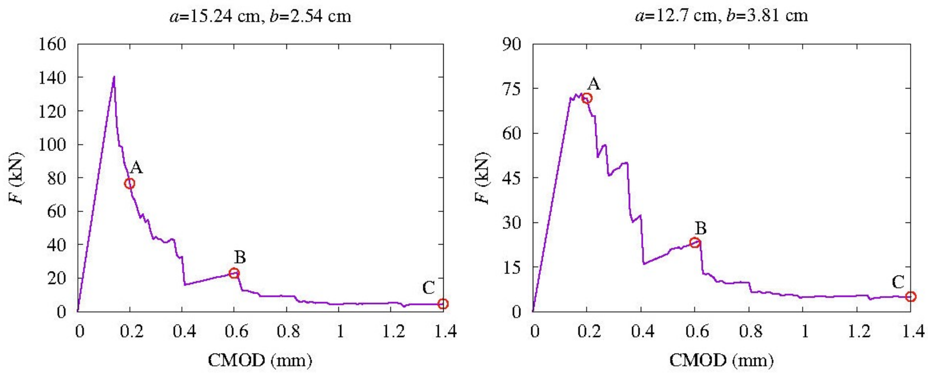

Fig. 9. Bending beam with three holes: force-displacement curves

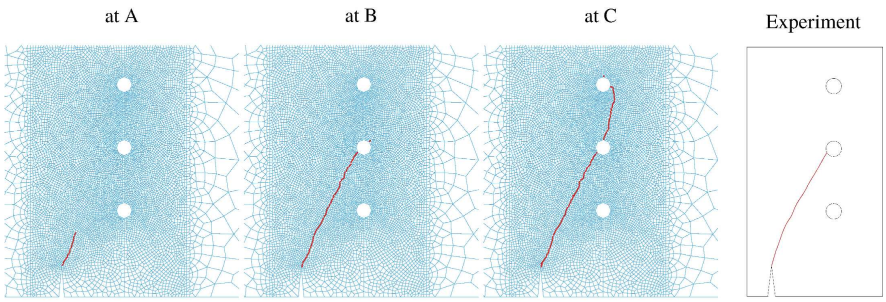

Fig. 10. Bending beam with three holes: cracked elements with a=15.24 cm, b=2.54 cm, comparing to the experimental result provided in [134]

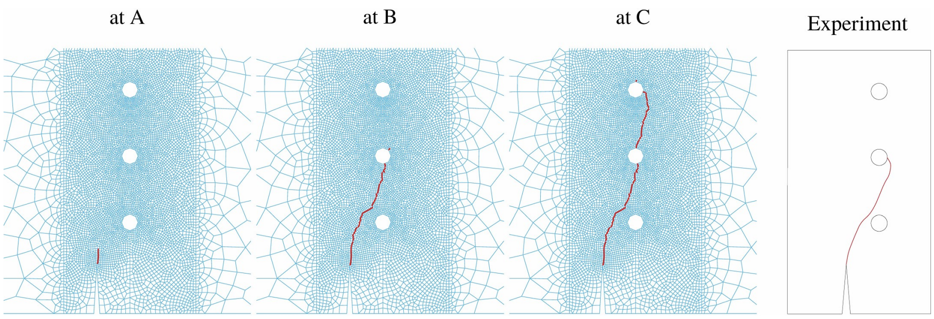

Fig. 11. Bending beam with three holes: crack paths with a = 12.7 cm, b = 3.81 cm, comparing to the experimental result provided in [134]

4. Conclusions

In this work, the numerical procedure of the CEM method was provided in detail, with its local criteria for crack growths. The advantages of CEM were shown that it does not need remeshing technique, cover algorithm, nodal enrichment or specific crack tracking strategy, which is built in the framework of conventional FEM with the standard Galerkin method. The numerical capability of CEM was investigated with several benchmark tests with or without initial imperfections, considering different discretizations and model set-ups. The results further proved the reliability and effectiveness of the CEM method, naturally capturing the initiation and propagation of cracks, which is capable of simulating random distributed cracks.

Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The authors gratefully acknowledge the financial support offered by NSFC 51809069, NSFC 11772234 and Hebei key research and development program 18216110D.

References

[1] H. Horii and T. Ichinomiya, Observation of fracture process zone by laser speckle technique and governing mechanism in fracture of concrete, *International Journal of Fracture*, 51 (1991) 19-29.
[2] F. Wittmann and X. Hu, Fracture process zone in cementitious materials, *International Journal of Fracture*, 51 (1991) 3-18.
[3] R. de Borst, Some recent issues in computational failure mechanics, *International Journal for Numerical Methods in Engineering*, 52 (2001) 63-95.
[4] J. Oliver, M. Cervera, and O. Manzoli, Strong discontinuities and continuum plasticity models: the strong discontinuity approach, *International Journal of Plasticity*, 15 (1999) 319-351.
[5] M. Cervera, M. Chiumenti, and D. D. Capua, Benchmarking on bifurcation and localization in J2 plasticity for plane stress and plane strain conditions, *Computer Methods in Applied Mechanics and Engineering*, 241-244 (2012) 206-224.
[6] J.-Y. Wu and M. Cervera, On the equivalence between traction- and stress-based approaches for the modeling of localized failure in solids, *Journal of the Mechanics and Physics of Solids*, 82 (2015) 137-163.
[7] L. Qing and Q. Li, A theoretical method for determining initiation toughness based on experimental peak load, *Engineering Fracture Mechanics*, 99 (2013) 295-305.
[8] L. Qing, M. Dong, and J. Guan, Determining initial fracture toughness of concrete for split-tension specimens based on the extreme theory, *Engineering Fracture Mechanics*, 189 (2018) 427-438.
[9] P. Areias, T. Rabczuk, and D. Dias-da-Costa, Element-wise fracture algorithm based on rotation of edges, *Engineering Fracture Mechanics*, 110 (2013) 113-137.
[10] B. Schrefler, S. Secchi, and L. Simoni, On adaptive refinement techniques in multifield problems including cohesive fracture, *Computer Methods in Applied Mechanics and Engineering*, 195 (2006) 444-461.
[11] P. Areias and T. Rabczuk, Steiner-point free edge cutting of tetrahedral meshes with applications in fracture, *Finite Elements in Analysis and Design*, 132 (2017) 27-41.
[12] X. Ren and J. Li, Dynamic fracture in irregularly structured systems, *Physical Review E*, 85 (2012) 055102.
[13] I. Khisamitov and G. Meschke, Variational approach to interface element modeling of brittle fracture propagation, *Computer Methods in Applied Mechanics and Engineering*, 328 (2018) 452-476.
[14] M. Cervera, M. Chiumenti, and R. Codina, Mixed stabilized finite element methods in nonlinear solid mechanics: Part I: Formulation, *Computer Methods in Applied Mechanics and Engineering*, 199 (2010) 2559-2570.
[15] M. Cervera, M. Chiumenti, and R. Codina, Mesh objective modeling of cracks using continuous linear strain and displacement interpolations, *International Journal for Numerical Methods in Engineering*, 87 (2011) 962-987.
[16] M. Cervera, G. Barbat, and M. Chiumenti, Finite element modeling of quasi-brittle cracks in 2d and 3d with enhanced strain accuracy, *Computational Mechanics*, 60 (2017) 767-796.
[17] C. Miehe, M. Hofacker, and F. Welschinger, A phase field model for rate independent crack propagation: Robust algorithmic implementation based on operator splits, *Computer Methods in Applied Mechanics and Engineering*, 199 (2010) 2765-2778.
[18] H. Badnava, M. Msekh, E. Etemadi, and T. Rabczuk, An h-adaptive thermomechanical phase field model for fracture, *Finite Elements in Analysis and Design*, 138 (2018) 31-47.
[19] J.-Y. Wu, A unified phase-field theory for the mechanics of damage and quasi-brittle failure, *Journal of the Mechanics and Physics of Solids*, 103 (2017) 72-99.
[20] J.-Y. Wu, A geometrically regularized gradient-damage model with energetic equivalence, *Computer Methods in Applied Mechanics and Engineering*, 328 (2018) 612- 637.
[21] J.-Y. Wu and M. Cervera, A thermodynamically consistent plastic-damage framework for localized failure in quasi-brittle solids: Material model and strain localization analysis, *International Journal of Solids and Structures*, 88-89 (2016) 227-247.
[22] P. Areias, T. Rabczuk, and M. Msekh, Phase-field analysis of finite-strain plates and shells including element subdivision,

Computer Methods in Applied Mechanics and Engineering, 312 (2016) 322-350.
[23] F. Amiri, D. Mill´an, Y. Shen, T. Rabczuk, and M. Arroyo, Phase-field modeling of fracture in linear thin shells, T*heoretical and Applied Fracture Mechanics*, 69 (2014) 102-109.
[24] J. Wu, C. McAuliffe, H. Waisman, and G. Deodatis, Stochastic analysis of polymer composites rupture at large deformations modeled by a phase field method, *Computer Methods in Applied Mechanics and Engineering*, 312 (2016) 596-634.
[25] S. Zhou, X. Zhuang, and T. Rabczuk, A phase-field modeling approach of fracture propagation in poroelastic media, *Engineering Geology*, 240 (2018) 189-203.
[26] S. Zhou, X. Zhuang, H. Zhu, and T. Rabczuk, Phase field modelling of crack propagation, branching and coalescence in rocks, *Theoretical and Applied Fracture Mechanics*, 96 (2018) 174-192.
[27] J.-Y. Wu and V. P. Nguyen, A length scale insensitive phase-field damage model for brittle fracture, *Journal of the Mechanics and Physics of Solids*, 119 (2018) 20-42.
[28] J.-Y. Wu, Robust numerical implementation of non-standard phase-field damage models for failure in solids, *Computer Methods in Applied Mechanics and Engineering*, 340 (2018) 767 - 797.
[29] D. D. da Costa, V. Cervenka, and R. G. e Costa, Model uncertainty in discrete and smeared crack prediction in rc beams under flexural loads, *Engineering Fracture Mechanics*, 199 (2018) 532 - 543.
[30] G. Ma, X. An, H. Zhang, and L. Li, Modeling complex crack problems using the numerical manifold method, *International Journal of Fracture*, 156 (2009) 21-35.
[31] Y. Ning, Z. Yang, B. Wei, and B. Gu, Advances in two-dimensional discontinuous deformation analysis for rock-mass dynamics, *International Journal of Geomechanics*, 17 (2017) E6016001.
[32] H. Zheng and D. Xu, New strategies for some issues of numerical manifold method in simulation of crack propagation, *International Journal for Numerical Methods in Engineering*, 97 (2014) 986-1010.
[33] H. Zheng, F. Liu, and X. Du, Complementarity problem arising from static growth of multiple cracks and MLS-based numerical manifold method, *Computer Methods in Applied Mechanics and Engineering*, 295 (2015) 150-171.
[34] J. Wu and Y. Cai, A partition of unity formulation referring to the NMM for multiple intersecting crack analysis, *Theoretical and Applied Fracture Mechanics*, 72 (2014) 28-36.
[35] H. Fan, H. Zheng, C. Li, and S. He, A decomposition technique of generalized degrees of freedom for mixed mode crack problems, *International Journal for Numerical Methods in Engineering*, 112(7) (2017) 803-831.
[36] J. Oliver, I. Dias, and A. Huespe, Crack-path field and strain-injection techniques in computational modeling of propagating material failure, *Computer Methods in Applied Mechanics and Engineering*, 274 (2014) 289-348.
[37] C. Linder and F. Armero, Finite elements with embedded branching, *Finite Elements in Analysis and Design*, 45 (2009) 280-293.
[38] M. Motamedi, D. Weed, and C. Foster, Numerical simulation of mixed mode (I and II) fracture behavior of pre-cracked rock using the strong discontinuity approach, *International Journal of Solids and Structures*, 85-86 (2016) 44-56.
[39] A. Tabarraei, J.-H. Song, and H. Waisman, A two-scale strong discontinuity approach for evolution of shear bands under dynamic impact loads, *International Journal for Multiscale Computational Engineering*, 2013 (2013) 543-563.
[40] E. Kishta, C. Giry, B. Richard, F. Ragueneau, and M. Balmaseda, A discrete anisotropic damage constitutive law with an enhanced mixed-mode kinematics: Application to RC shear walls, *Engineering Fracture Mechanics*, 184 (2017) 121-140.
[41] T. C. Gasser and G. A. Holzapfel, Geometrically non-linear and consistently linearized embedded strong discontinuity models for 3D problems with an application to the dissection analysis of soft biological tissues, *Computer Methods in Applied Mechanics and Engineering*, 192 (2003) 5059-5098.
[42] Z. Zhang, D. Wang, and X. Ge, A novel triangular finite element partition method for fracture simulation without enrichment of interpolation, *International Journal of Computational Methods*, 10 (2013) 1350015.
[43] Z. Zhang, D. Wang, X. Ge, and H. Zheng, Three-dimensional element partition method for fracture simulation, *International Journal of Geomechanics*, 16 (2016) 04015074.
[44] S. Saloustros, L. Pel`a, M. Cervera, and P. Roca, Finite element modelling of internal and multiple localized cracks, *Computational Mechanics*, 59 (2017) 299-316.
[45] S. Saloustros, M. Cervera, and L. Pel`a, Tracking multi-directional intersecting cracks in numerical modelling of masonry shear walls under cyclic loading, *Meccanica*, 53(7) (2018) 1757-1776.
[46] M. Nikoli´c, A. Ibrahimbegovic, and P. Miscevic, Discrete element model for the analysis of fluid-saturated fractured poro-plastic medium based on sharp crack representation with embedded strong discontinuities, *Computer Methods in Applied Mechanics and Engineering*, 298 (2016) 407-427.
[47] M. Nikoli´c, X. N. Do, A. Ibrahimbegovic, and Z. Nikoli´c, Crack propagation in dynamics by embedded strong discontinuity approach: Enhanced solid versus discrete lattice model, *Computer Methods in Applied Mechanics and Engineering*, 340 (2018) 480 - 499.
[48] J.-Y. Wu and F.-B. Li, An improved stable XFEM (Is-XFEM) with a novel enrichment function for the computational modeling of cohesive cracks, *Computer Methods in Applied Mechanics and Engineering*, 295 (2015) 77-107.
[49] J.-H. Song, P. Areias, and T. Belytschko, A method for dynamic crack and shear band propagation with phantom nodes, *International Journal for Numerical Methods in Engineering*, 67 (2006) 868-893.
[50] T. Chau-Dinh, G. Zi, P.-S. Lee, T. Rabczuk, and J.-H. Song, Phantom-node method for shell models with arbitrary cracks, *Computers and Structures*, 92-93 (2012) 242-246.
[51] S. Ghorashi, N. Valizadeh, S. Mohammadi, and T. Rabczuk, T-spline based XIGA for fracture analysis of orthotropic

media, *Computers & Structures*, 147 (2015) 138-146.
[52] T. C. Gasser and G. A. Holzapfel, Modeling 3D crack propagation in unreinforced concrete using PUFEM, *Computer Methods in Applied Mechanics and Engineering*, 194 (2005) 2859-2896.
[53] Y. Wang and H. Waisman, From diffuse damage to sharp cohesive cracks: A coupled XFEM framework for failure analysis of quasi-brittle materials, *Computer Methods in Applied Mechanics and Engineering*, 299 (2016) 57-89.
[54] Y. Wang, H. Waisman, and I. Harari, Direct evaluation of stress intensity factors for curved cracks using Irwin's integral and XFEM with high-order enrichment functions, *International Journal for Numerical Methods in Engineering*, 112(7) (2017) 629-654.
[55] Y. Wang and H. Waisman, Material-dependent crack-tip enrichment functions in XFEM for modeling interfacial cracks in bimaterials, *International Journal for Numerical Methods in Engineering*, 112(11) (2017) 1495-1518.
[56] Y. Wang, C. Cerigato, H. Waisman, and E. Benvenuti, XFEM with high-order material-dependent enrichment functions for stress intensity factors calculation of interface cracks using Irwin's crack closure integral, *Engineering Fracture Mechanics*, 178 (2017) 148-168.
[57] T. Rabczuk and T. Belytschko, Adaptivity for structured meshfree particle methods in 2D and 3D, *International Journal for Numerical Methods in Engineering*, 63 (2005) 1559-1582.
[58] X. Zhuang, C. Augarde, and K. Mathisen, Fracture modeling using meshless methods and level sets in 3D: framework and modeling, *International Journal for Numerical Methods in Engineering*, 92 (2012) 969-998.
[59] X. Zhuang, C. Augarde, and S. Bordas, Accurate fracture modelling using meshless methods, the visibility criterion and level sets: Formulation and 2D modelling, *International Journal for Numerical Methods in Engineering*, 86 (2011) 249-268.
[60] T. Rabczuk, P. Areias, and T. Belytschko, A meshfree thin shell method for nonlinear dynamic fracture, International *Journal for Numerical Methods in Engineering*, 72 (2007) 524-548.
[61] T. Rabczuk, S. Bordas, and G. Zi, A three-dimensional meshfree method for continuous multiple-crack initiation, propagation and junction in statics and dynamics, *Computational Mechanics*, 40 (2007) 473-495.
[62] T. Rabczuk and G. Zi, A meshfree method based on the local partition of unity for cohesive cracks, *Computational Mechanics*, 39 (2007) 743-760.
[63] S. Bordas, T. Rabczuk, and G. Zi, Three-dimensional crack initiation, propagation, branching and junction in non-linear materials by an extended meshfree method without asymptotic enrichment, *Engineering Fracture Mechanics*, 75 (2008) 943-960.
[64] T. Rabczuk, S. Bordas, and G. Zi, On three-dimensional modelling of crack growth using partition of unity methods, *Computers and Structures*, 88 (2010) 1391-1411.
[65] Y. Cai, P. Sun, H. Zhu, and T. Rabczuk, A mixed cover meshless method for elasticity and fracture problems, *Theoretical and Applied Fracture Mechanics*, 95 (2018) 73-103.
[66] Z.-J. Fu, W. Chen, and H.-T. Yang, Boundary particle method for Laplace transformed time fractional diffusion equations, *Journal of Computational Physics*, 235 (2013) 52-66.
[67] H. Zhu, X. Zhuang, and Y. Cai, High rock slope stability analysis using the enriched meshless shepard and least squares method, *International Journal of Computational Methods*, 8 (2011) 209-228.
[68] T. Rabczuk and T. Belytschko, Cracking particles: a simplified meshfree method for arbitrary evolving cracks, *International Journal for Numerical Methods in Engineering*, 61 (2004) 2316-2343.
[69] T. Rabczuk and T. Belytschko, A three-dimensional large deformation meshfree method for arbitrary evolving cracks, *Computer Methods in Applied Mechanics and Engineering*, 196 (2007) 2777-2799.
[70] T. Rabczuk, G. Zi, S. Bordas, and H. Nguyen-Xuan, A simple and robust three-dimensional cracking-particle method without enrichment, *Computer Methods in Applied Mechanics and Engineering*, 199 (2010) 2437-2455.
[71] W. Ai and C. Augarde, An adaptive cracking particle method for 2d crack propagation, *International Journal for Numerical Methods in Engineering*, 108(13) (2016) 1626-1648.
[72] S. Silling, Reformulation of elasticity theory for discontinuities and long-range force, *Journal of the Mechanics and Physics of Solids*, 48 (2000) 175-209.
[73] R. Macek and S. Silling, Peridynamics via finite element analysis, *Finite Elements in Analysis and Design*, 43 (2007) 1169-1178.
[74] S. Silling and E. Askari, A meshfree method based on the peridynamic model of solid mechanics, *Computers and Structures*, 83 (2005) 1526-1535.
[75] H. Ren, X. Zhuang, and T. Rabczuk, Dual-horizon peridynamics: A stable solution to varying horizons, *Computer Methods in Applied Mechanics and Engineering,* 318 (2017) 762-782.
[76] H. Ren, X. Zhuang, and T. Rabczuk, Dual-horizon peridynamics: A stable solution to varying horizons, *Computer Methods in Applied Mechanics and Engineering*, 318 (2017) 762-782.
[77] S. N. Butt, J. J. Timothy, and G. Meschke, Wave dispersion and propagation in state-based peridynamics, *Computational Mechanics*, 60(5) (2017) 725-738.
[78] Y. D. Ha and F. Bobaru, Studies of dynamic crack propagation and crack branching with peridynamics, *International Journal of Fracture*, 162 (2010) 229-244.
[79] T. Rabczuk and H. Ren, A peridynamics formulation for quasi-static fracture and contact in rock, *Engineering Geology*, 225 (2017) 42-48.
[80] F. Han, G. Lubineau, Y. Azdoud, and A. Askari, A morphing approach to couple state-based peridynamics with classical continuum mechanics, *Computer Methods in Applied Mechanics and Engineering*, 301 (2016) 336 - 358.

[81] F. Han, G. Lubineau, and Y. Azdoud, Adaptive coupling between damage mechanics and peridynamics: A route for objective simulation of material degradation up to complete failure, *Journal of the Mechanics and Physics of Solids*, 94 (2016) 453-472.
[82] Y. Bie, X. Cui, and Z. Li, A coupling approach of state-based peridynamics with node-based smoothed finite element method, *Computer Methods in Applied Mechanics and Engineering*, 331 (2018) 675 - 700.
[83] H. Fei and L. Gilles, Coupling of nonlocal and local continuum models by the Arlequin approach, *International Journal for Numerical Methods in Engineering*, 89(6) (2011) 671-685.
[84] M. Cervera, L. Pela, R. Clemente, and P. Roca, A crack-tracking technique for localized damage in quasi-brittle materials, *Engineering Fracture Mechanics*, 77 (2010) 2431-2450.
[85] P. Dumstorff and G. Meschke, Crack propagation criteria in the framework of XFEM-based structural analyses, *International Journal for Numerical and Analytical Methods in Geomechanics*, 31 (2007) 239-259.
[86] A. Alsahly, C. Callari, and G. Meschke, An algorithm based on incompatible modes for the global tracking of strong discontinuities in shear localization analyses, *Computer Methods in Applied Mechanics and Engineering*, 330 (2018) 33-63.
[87] M. Stolarska, D. Chopp, N. Mo¨es, and T. Belytschko, Modelling crack growth by level sets in the extended finite element method, *International Journal for Numerical Methods in Engineering*, 51 (2001) 943-960.
[88] F. Riccardi, E. Kishta, and B. Richard, A step-by-step global crack-tracking approach in E-FEM simulations of quasi-brittle materials, *Engineering Fracture Mechanics*, 170 (2017) 44-58.
[89] C. Annavarapu, R. R. Settgast, E. Vitali, and J. P. Morris, A local crack-tracking strategy to model three-dimensional crack propagation with embedded methods, *Computer Methods in Applied Mechanics and Engineering*, 311 (2016) 815-837.
[90] S. Saloustros, M. Cervera, and L. Pel`a, Challenges, tools and applications of tracking algorithms in the numerical modelling of cracks in concrete and masonry structures, *Archives of Computational Methods in Engineering*, 2018. doi:10.1007/s11831-018- 9274-3.
[91] Y. Zhang and X. Zhuang, Cracking elements: a self-propagating strong discontinuity embedded approach for quasi-brittle fracture, *Finite Elements in Analysis and Design*, 144 (2018) 84-100.
[92] T. Belytschko, J. Fish, and B. E. Engelmann, A Finite element with embedded localization zones, *Computer Methods in Applied Mechanics and Engineering*, 70 (1988) 59-89.
[93] Y. Zhang and X. Zhuang, Cracking elements method for dynamic brittle fracture, *Theoretical and Applied Fracture Mechanics*, 102 (2019) 1-9
[94] R. Larsson, K. Runesson, and S. Sture, Embedded localization band in undrained soil based on regularized strong discontinuity-theory and FE-analysis, *International Journal of Solids and Structures*, 33 (1996) 3081-3101.
[95] J. Oliver, A. Huespe, and E. Samaniego, A study on finite elements for capturing strong discontinuities, *International Journal for Numerical Methods in Engineering*, 56 (2003) 2135-2161.
[96] E. Samaniego, Contributions to the continuum modelling of strong discontinuities in two-dimensional solids. *PhD thesis*, Polytechnic University of Catalonia, 2003.
[97] Y. Zhang, R. Lackner, M. Zeiml, and H. Mang, Strong discontinuity embedded approach with standard SOS formulation: Element formulation, energy-based crack tracking strategy, and validations, *Computer Methods in Applied Mechanics and Engineering*, 287 (2015) 335-366.
[98] G. Camacho and M. Ortiz, Computational modelling of impact damage in brittle materials, *International Journal of Solids and Structures*, 33 (1996) 2899-2938.
[99] G. Meschke and P. Dumstorff, Energy-based modeling of cohesive and cohesionless cracks via X-FEM, *Computer Methods in Applied Mechanics and Engineering*, 196 (2007) 2338-2357.
[100] S. Mariani and U. Perego, Extended finite element method for quasi-brittle fracture, *International Journal for Numerical Methods in Engineering*, 58 (2003) 103-126.
[101] T. Belytschko, D. Organ, and C. Gerlach, Element-free Galerkin methods for dynamic fracture in concrete, *Computer Methods in Applied Mechanics and Engineering*, 187 (2000) 385-399.
[102] Y. Zhang and X. Zhuang, Cracking elements method for dynamic brittle fracture, *Theoretical and Applied Fracture Mechanics*, 102 (2019) 1-9.
[103] Y. Zhang and X. Zhuang, A softening-healing law for self-healing quasi-brittle materials: analyzing with strong discontinuity embedded approach, *Engineering Fracture Mechanics*, 192 (2018) 290-306.
[104] J. Oliver, A consistent characteristic length for smeared cracking models, *International Journal for Numerical Methods in Engineering*, 28 (1989) 461-474.
[105] M. Jir´asek, Comparative study on finite elements with embedded discontinuities, *Computer Methods in Applied Mechanics and Engineering*, 188 (2000) 307-330.
[106] H. Lotfi and P. Shing, Embedded representation of fracture in concrete with mixed finite elements, *International Journal for Numerical Methods in Engineering*, 38 (1995) 1307-1325.
[107] E. N. Dvorkin, A. M. Cuiti˜no, and G. Gioia, Finite elements with displacement interpolated embedded localization lines insensitive to mesh size and distortions, *International Journal for Numerical Methods in Engineering*, 30 (1990) 541-564.
[108] J. Oliver, Modelling strong discontinuities in solid mechanics via strain softening constitutive equations. Part I: Fundamentals, *International Journal for Numerical Methods in Engineering*, 39 (1996) 3575-3600.
[109] J. Oliver, Modelling strong discontinuities in solid mechanics via strain softening constitutive equations. Part II: Numerical simulation, *International Journal for Numerical Methods in Engineering*, 39 (1996) 3601-3623.
[110] F. Armero and K. Garikipati, An analysis of strong discontinuities in multiplicative finite strain plasticity and their

relation with the numerical simulation of strain localization in solids, *International Journal of Solids and Structures*, 33 (1996) 2863-2885.

[111] J. Oliver, A. Huespe, M. Pulido, and E. Samaniego, On the strong discontinuity approach in finite deformation settings, *International Journal for Numerical Methods in Engineering*, 56 (2003) 1051-1082.

[112] R. I. Borja, A finite element model for strain localization analysis of strongly discontinuous fields based on standard Galerkin approximation, *Computer Methods in Applied Mechanics and Engineering*, 190 (2000) 1529-1549.

[113] R. I. Borja, Finite element simulation of strain localization with large deformation: capturing strong discontinuity using a Petrov-Galerkin multiscale formulation, *Computer Methods in Applied Mechanics and Engineering*, 191 (2002) 2949-2978.

[114] J. Oliver, A. Huespe, S. Blanco, and D. Linero, Stability and robustness issues in numerical modeling of material failure with the strong discontinuity approach, *Computer Methods in Applied Mechanics and Engineering*, 195 (2006) 7093-7114.

[115] J. Mosler and G. Meschke, 3D modelling of strong discontinuities in elastoplastic solids: fixed and rotating localization formulations, *International Journal for Numerical Methods in Engineering*, 57 (2003) 1553-1576.

[116] C. Feist and G. Hofstetter, An embedded strong discontinuity model for cracking of plain concrete, *Computer Methods in Applied Mechanics and Engineering*, 195 (2006) 7115-7138.

[117] C. Feist and G. Hofstetter, Three-dimensional fracture simulations based on the SDA, *International Journal for Numerical and Analytical Methods in Geomechanics*, 31 (2007) 189-212.

[118] J. Simo and T. Hughes, *Computational Inelasticity*. Springer, 1998.

[119] X. Ren and J. Li, Two-level consistent secant operators for cyclic loading of structures, *Journal of Engineering Mechanics*, 2018. DOI: 10.1061/(ASCE)EM.1943-7889.0001494.

[120] K.-J. Bathe, *Finite Element Procedures (2nd edition)*. 2014. ISBN-10: 0979004950.

[121] Y. Zhang, C. Pichler, Y. Yuan, M. Zeiml, and R. Lackner, Micromechanics-based multifield framework for early-age concrete, *Engineering Structures*, 47 (2013) 16-24.

[122] Y. Zhang, M. Zeiml, C. Pichler, and R. Lackner, Model-based risk assessment of concrete spalling in tunnel linings under fire loading, *Engineering Structures*, 77 (2014) 207-215.

[123] Y. Zhang, M. Zeiml, M. Maier, Y. Yuan, and R. Lackner, Fast assessing spalling risk of tunnel linings under RABT fire: From a coupled thermo-hydro-chemo-mechanical model towards an estimation method, *Engineering Structures*, 142 (2017) 1-19.

[124] Y. Zhang, Multi-slicing strategy for the three-dimensional discontinuity layout optimization (3D DLO), *International Journal for Numerical and Analytical Methods in Geomechanics*, 41 (2017) 488-507.

[125] Y. Zhang and X. Zhuang, Stability analysis of shotcrete supported crown of NATM tunnels with discontinuity layout optimization, *International Journal for Numerical and Analytical Methods in Geomechanics*, 42 (2018) 1199-1216.

[126] C. Xiong, L. Cheng, F. Tong, and H. An, Oscillatory flow regimes for a circular cylinder near a plane boundary, *Journal of Fluid Mechanics*, 844 (2018) 127-161.

[127] Z. Fu, W. Chen, P. Wen, and C. Zhang, Singular boundary method for wave propagation analysis in periodic structures, *Journal of Sound and Vibration*, 425 (2018) 170-188.

[128] S. Lin, Y. Xie, Q. Li, X. Huang, and S. Zhou, On the shape transformation of cone scales, *Soft Matter.*, 12 (2016) 9797-9802.

[129] Z. Sun, X. Yan, R. Liu, Z. Xu, S. Li and Y. Zhang, Transient analysis of grout penetration with time-dependent viscosity inside 3D fractured rock mass by Unified Pipe-Network Method, *Water*, 10 (2018) ID 1122, w10091122

[130] Z. Sun, X. Yan, W. Han, G. Ma and Y. Zhang, Simulating the filtration effects of cement-grout in fractured porous media with the 3D Unified Pipe-Network Method, *Processes*, 7 (2019), ID 46, pr7010046

[131] N. A. Al-Shayea, Crack propagation trajectories for rocks under mixed mode i-ii fracture, *Engineering Geology*, 81(1) (2005) 84-97.

[132] P. Areias, M. Msekh, and T. Rabczuk, Damage and fracture algorithm using the screened Poisson equation and local remeshing, *Engineering Fracture Mechanics*, 158 (2016) 116-143.

[133] M. Ambati, T. Gerasimov, and L. De Lorenzis, A review on phase-field models of brittle fracture and a new fast hybrid formulation, *Computational Mechanics*, 55 (2015) 383-405.

[134] T. Bittencourt, P. Wawrzynek, A. Ingraffea, and J. Sousa, Quasi-automatic simulation of crack propagation for 2d lefm problems, *Engineering Fracture Mechanics*, 55(2) (1996) 321-334.

[135] A. Mesgarnejad, B. Bourdin, and M. Khonsari, Validation simulations for the variational approach to fracture, *Computer Methods in Applied Mechanics and Engineering*, 290 (2015) 420 - 437.

[136] P. Areias, T. Rabczuk, and D. Dias-da-Costa, Element-wise fracture algorithm based on rotation of edges, *Engineering Fracture Mechanics*, 110 (2013) 113-137.

[137] Y. Wang and H. Waisman, An arc-length method for controlled cohesive crack propagation using high-order XFEM and irwins crack closure integral, *Engineering Fracture Mechanics*, 199 (2018) 235-256.

[138] Y. Wang and H. Waisman, Progressive delamination analysis of composite materials using XFEM and a discrete damage zone model, *Computational Mechanics*, 55(1) (2015) 1-26.

[139] S. May, J. Vignollet, and R. de Borst, A new arc-length control method based on the rates of the internal and the dissipated energy, *Engineering Computations*, 33(1) (2016) 100-115.