Poiseuille Flow with Couple Stresses Effect and No-slip Boundary Conditions

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Abstract. In this paper, the problem of Poiseuille flow with couple stresses effect in a fluid layer using the linear instability and nonlinear stability theories is analyzed. Also, the nonlinear stability eigenvalue problems for \(x,z\) and \(y,z\) disturbances are derived. The Chebyshev collocation method is adopted to arrive at the eigenvalue equation, which is then solved numerically, where the equivalent of the Orr-Sommerfeld eigenvalue problem is solved using the Chebyshev collocation method. The difficulties which arise in computing the spectrum of the Orr-Sommerfeld equation are discussed. The critical Reynolds number \(R_c\), the critical wave number \(a_c\), and the critical wave speed \(c_c\) are computed for wide ranges of the couple stresses coefficient \(M\). It is found that the couple stresses coefficient \(M\) has great stabilizing effects on the fluid flow where the fluid flow becomes more unstable as \(M\) increases.

Keywords: Poiseuille flow, Couple stresses, Orr-Sommerfeld, Linear instability, Nonlinear stability.

1. Introduction

In this paper, we study the problem of Poiseuille flow. It involves an incompressible fluid under isothermal conditions, inside an infinite channel, which associates with a constant pressure gradient. In a laminar way, the fluid flows along this pressure gradient, resulting in a parabolic velocity profile which is independent of time. The actual difference from the previous work is that we add the effect of the couple stresses effect.

One of the main problems in fluid dynamics is the classical hydrodynamic problem of stability for Poiseuille flow in a channel, see for example Joseph [1], chapter 3, Straughan [2], chapter 8. These authors have shown that there are substantial problems in trying to improve or develop the nonlinear stability theory because there is a significant difference between the linear instability and the nonlinear energy thresholds. In addition, the associated eigenvalue problems to this class of flows are very difficult and need a very accurate numerical scheme, see for example [3, 4].

However, this area still attracts much interest in the literature of fluid dynamics. In [5], the nonlinear energy stability has been addressed. Poiseuille flows that time-dependent and time-periodic was analyzed in [6, 7]. The effect of slip boundary conditions has been studied in [8, 9]. Moreover, the Poiseuille flow problem of flow for a fluid overlying a porous medium has been introduced in [10, 11]. Also, the Poiseuille problem of flow in a channel with one fluid overlying another was studied in [4]. These articles include many other relevant references.

Earlier in this work, the theoretical and experimental results on the onset of thermal instability (Bénard convection) in a fluid layer under varying assumptions of hydrodynamics, underwent a detailed review, conducted by Chandrasekhar [12]. Such investigations of these kinds of fluids are important, bearing in mind the increasing importance of non-Newtonian fluids in technology and industries. Stokes [13] has put forward the theory of couple-stress fluids. Couple-stresses are
present in significant magnitude in fluids with very large molecules. Applications of couple-stress fluids occur in
connection with the study of the mechanism of synovial joint lubrication, currently being focussed upon by researchers.
A human joint is a dynamically loaded bearing with articular cartilage as the bearing, and synovial fluid as the lubricant.
The normal synovial fluid is clear or yellowish and is a non-Newtonian, viscous fluid. Because of the long chain of
lauronic acid molecules found as additives in synovial fluid, Walicki and Walicka [14] modeled the fluid in question as
couple-stress fluid in human joints. The issue of a couple-stress fluid and porous medium has also been investigated in
[15-17]. Recently, in Harfash and Meftenb [18], the problem of convective movement of a reacting solute in a viscous
incompressible fluid occupying a plane layer and subjected to a couple stresses effects have been studied. The thresholds
for linear instability are found and compared to those derived by a global nonlinear energy stability analysis. However
Harfash and Meftenb, in [19], have extended their work where they have studied the problem of double-diffusive
convection in a reacting fluid with the effect of couple stresses. In [19], the density is assumed to have quadratic
dependence on the temperature and a linear dependence on the concentration and Linear instability and nonlinear
stability analyses were performed.
In the present article, the problem of Poiseuille flow of an incompressible couple stress fluid between parallel plates is
studied using no-slip boundary conditions. Here, we consider a modification to the work in [20] by inserting the effect of
couple stresses effect on the problem of Poiseuille flow in an infinite channel and discarding the effect of the magnetic
field and we believe this problem is analyzed for the first time in this article. In order to provide a clear analysis of the
problem of Orr-Sommerfeld, we discuss in this paper the instability of Poiseuille flow in the plane with the couple
stresses effect. We then turn to the study of non-linear stability analysis of the problem. We also include our
computational results for both cases.

2. Basic Equations

We suppose the Poiseuille flow with couple stresses type and occupies the spatial domain
\{(x, y) \in \mathbb{R}^2 \} \times \{z \in (-L/2, L/2) \} \times \{t > 0\}, (schematically shown in Figs. 1 and 2), then, the governing equations are
[13]:
\[
\rho_0(v_{i,j} + v_j v_{i,j}) = -p_{,j} + \mu \Delta v_i - \hat{\mu} \Delta^2 v_i, \\
v_{i,j} = 0,
\]
(1)
where \(v_i, p, \rho_0, \mu\), and \(\hat{\mu}\) denote the velocity field, pressure, density, dynamic viscosity coefficient, and couple stress
viscosity coefficient, see [18, 19]. The equations (1) are conveniently non-dimensionalized scalings with the variables:
\[
\begin{align*}
    x &= Lx' ,
    t &= \frac{L}{V_0}t' ,
    v &= v_0V' ,
    p &= \frac{\rho_0 \mu}{L}p' ,
    M &= \frac{\hat{\mu}}{\mu L^2} ,
    Re &= \frac{\rho_0V_0L}{\mu}.
\end{align*}
\]
Now, from above non-dimensional, then the system (1) may be rewritten as:
\[
Re(v_{i,j} + v_j v_{i,j}) = -p_{,j} + \Delta v_i - M \Delta^2 v_i, \\
v_{i,j} = 0,
\]
(2)
where \(Re\) is the Reynolds number and \(M\) is a non-dimensional couple stress viscosity coefficient. The above
Equations are holding now on \{(x, y) \in \mathbb{R}^2 \} \times \{z \in (-1,1) \} \times \{t > 0\}, with the following no-slip boundary conditions, see
[18, 19],
\[
w_{yz} - v_z = 0 \quad w_{zz} - u_{zz} = 0, \quad \text{and} \quad v_i = 0, \quad \text{on} \quad z = -1,1.
\]
(3)

Fig. 1. A schematic of the physical domain.
Poiseuille flow with couple stresses effect and no-slip boundary conditions

\[ w_{zz} - v_{zz} = 0, \quad w_{zz} - u_{zz} = 0 \quad \text{and} \quad v_i = 0 \quad \text{on} \quad z = 1 \]

\[ w_{zz} - v_{zz} = 0, \quad w_{zz} - u_{zz} = 0 \quad \text{and} \quad v_i = 0 \quad \text{on} \quad z = -1 \]

Fixed

\[ \frac{\partial \bar{p}}{\partial x} = k^2 > 0, \]

where an over bar denotes the basic state, then,

\[ \Delta \bar{p} - M \Delta \bar{v}_i = -k^2, \]

the basic velocity field corresponding to this pressure gradient has form \( \bar{v}_i = (U(z), 0, 0) \), such that

\[ \Delta \bar{v}_i = (U^{''}(z), 0, 0) \text{ and } \Delta^2 \bar{v}_i = (U^{(4)}(z), 0, 0). \]

Under these treatments, we have:

\[ U^{''} - MU^{(4)} = -k^2, \]

where \( U^{''} \) and \( U^{(4)} \) are the second and fourth derivatives of \( U \), respectively. The boundary conditions are:

\[ U(z) = 0, \quad z = \pm 1, \]

\[ U^{''}(z) = 0, \quad z = \pm 1, \]

By solving the above equation and applying the boundary conditions we arrive at:

\[ U(z) = \frac{k^2 M}{\cosh\left(\frac{z}{\sqrt{M}}\right)} \cosh\left(\frac{z}{\sqrt{M}}\right) - k^2 M + \frac{k^2}{2} (1 - z^2), \]

if in our non-dimensionalization we take \( k^2 = 2 \), then \( U \) reduce to:

\[ U(z) = \frac{2 M}{\cosh\left(\frac{z}{\sqrt{M}}\right)} \cosh\left(\frac{z}{\sqrt{M}}\right) - 2 M + 1 - z^2. \]

The perturbation forms are:

\[ v_i = U + u_i \quad \text{and} \quad p = \bar{p} + \pi. \]

Substituting the perturbed forms into equations (2), we obtain the perturbation equations:

\[ Re(u_{i,j} + u_iu_{j,i} + u_jU_{i,j} + U_j u_{i,j}) = -\pi_{ij} + \Delta u_i - M \Delta^2 u_i, \]

\[ u_{i,j} = 0. \]
### 3. Linear Instability

To study linear instability, we ignore nonlinear terms in (4), then we obtain:

\[
Re(u_{ix} + u_j U_{ij} + U_{ij} u_{ij}) = -\pi_j + \Delta u_i - M\Delta u_i, \\
\quad u_{ij} = 0,
\]

since \(u_i = (u, v, w)\), then, the above equations can be written as the form:

\[
\begin{align*}
Re(u_{ij} + wU_{ij} + U_{ij} u_{ij}) &= -\pi_j + \Delta u_i - M\Delta u_i, \\
Re(v_{ij} + U_{ij} v_{ij}) &= -\pi_j + \Delta v - M\Delta v, \\
Re(w_{ij} + U_{ij} w_{ij}) &= -\pi_j + \Delta w - M\Delta^2 w, \\
\quad u_i + v_i + w_i = 0.
\end{align*}
\]

We consider a solution forms:

\[
\begin{align*}
u &= u(z)e^{i(\alpha x + \beta y - \omega t)}, \\
v &= v(z)e^{i(\alpha x + \beta y - \omega t)}, \\
\pi &= \pi(z)e^{i(\alpha x + \beta y - \omega t)}.
\end{align*}
\]

Then, by substituting equations (7) in equations (6), yields:

\[
\begin{align*}
aRe(U - c)u + RU'w &= -i\alpha\pi + (D^2 - \alpha^2 - \beta^2)u - M(D^2 - \alpha^2 - \beta^2)u, \\
aRe(U - c)v &= -i\beta\pi + (D^2 - \alpha^2 - \beta^2)v - M(D^2 - \alpha^2 - \beta^2)v, \\
aRe(U - c)w &= -D\pi + (D^2 - \alpha^2 - \beta^2)w - M(D^2 - \alpha^2 - \beta^2)w, \\
a\alpha u + i\beta v + Dw &= 0.
\end{align*}
\]

However, if we add \(\alpha\times(7)_1\) to \(\beta\times(7)_2\), and define:

\[
a = \sqrt{\alpha^2 + \beta^2}, \quad a\alpha u + \beta v, \quad \hat{w} = w, \quad a\Re a\Re = a\Re, \quad a\pi = a\pi,
\]
we obtain the system:

\[
\begin{align*}
a^2 \hat{\Re}e(U - c)\hat{u} + a\Re eU'\hat{w} &= -i\alpha\hat{\pi} + a(D^2 - \alpha^2)\hat{u} - aM(D^2 - \alpha^2)\hat{u}, \\
a\Re eU - c\Re w &= -D\hat{\pi} + (D^2 - \alpha^2)\hat{w} - M(D^2 - \alpha^2)\hat{w}, \\
\quad i\alpha\hat{u} + i\beta \hat{v} + D\hat{w} &= 0.
\end{align*}
\]

since \(i\alpha\hat{u} = -D\hat{w}\), then, equation (9), reduces to:

\[
-a\Re e(U - c)\Re w + a\Re eU'\hat{w} = -i\alpha\hat{\pi} + i(D^2 - \alpha^2)D\hat{w} - iM(D^2 - \alpha^2)^2 D\hat{w}.
\]

Now, remove the pressure terms by performing \(D \times (10) - i\alpha^2 \times (109)_2\), yields:

\[
\begin{align*}
\quad i\Re e[(U - c)(D^2 - \alpha^2) - U']w &= [(D^2 - \alpha^2)\hat{w} - M(D^2 - \alpha^2)^2]w.
\end{align*}
\]

where \(z \in (-1, 1)\), with the following boundary conditions:

\[
W = DW = D^3W = 0, \quad z = \pm 1.
\]

### 4. Nonlinear Theory

We use now the energy method to derive sufficient conditions to ensure the stability of the steady-state solution, i.e., we will find value \(Re_e\) so that the steady-state solution is stable \(Re < Re_e\). Let \(\|\cdot\|\) and are the norm and inner product on \(L^2(\Omega)\), which have the form:

\[
\langle f, g \rangle = \int f \overline{g} \, dV, \quad \| f \| = \sqrt{\langle f, f \rangle}, \quad f, g \in L^2(\Omega),
\]

where \(dV = dx dy dz\) is a volume element and let \(H^1(\Omega)\) be the complex Hilbert space of measurable functions which is defined on \(\Omega\) such that for \(f \in H^1(\Omega)\), we have:

\[
\| f \| + \| f' \| < \infty.
\]
We also define the subspace $\mathcal{H} \in [H^1(\Omega)]^3$, where for $u = (u,v,w) \in \mathcal{H}$, then, $\nabla \cdot u = 0$, the components of $u$ satisfying (3) and is periodic modulo $\Omega$. Assuming $u \in \mathcal{H}$, and let $E(t)$ is defined by:

$$E(t) = \frac{1}{2} \| u \|^2$$

Now, we multiply the equation (4) by $u$, yields:

$$\frac{dE}{dt} = \langle u_j \left( \frac{1}{Re} \left[ -\Delta u_j + MA^2 u_j \right] - u_j u_{i,j} - u_j U_{i,j} - U_{j,i} u_{i,j} \right) \rangle,$$

integrating by parts for each of these terms and applying $u_{i,j} = 0$, we obtain:

$$\frac{dE}{dt} = -\frac{1}{2} \langle u_j (u_{i,j} + U_{i,j}) \rangle - \frac{1}{Re} \| \nabla u \|^2 + M \| \Delta u \|^2.$$

Now, let $F_y = \frac{1}{2} (U_{i,j} + U_{j,i})$, thus, the energy equation can be reduced as:

$$\frac{dE}{dt} = I - \frac{1}{Re} D,$$

where

$$I(u) = -\langle u_j, F_j \rangle \quad \text{and} \quad D(u) = \| \nabla u \|^2 + M \| \Delta u \|^2,$$

if we define $Re_k$ by:

$$\frac{1}{Re_k} = \max_{u \in \mathcal{H}} \frac{I}{D},$$

then with $\alpha$ being the constant in poincaré’s inequality for $u$, we have $D \geq \alpha E$, hence:

$$\frac{dE}{dt} \leq -\alpha \left( \frac{Re_k - Re}{ReRe_k} \right) E(t) \Rightarrow E(t) \leq e^{-\gamma t} E(0),$$

where $\gamma = \alpha \left( \frac{Re_k - Re}{ReRe_k} \right)$. The decay of $E(t)$, and hence of $u$ in an $L^2$ sense, follows from the above inequality.

Now that global stability has been satisfied and then it remains to solve the variational problem (14). To solve the maximization problem we study the Euler Lagrange equations. For an arbitrary function $h(u) \in \mathcal{H}$. Hence:

$$\frac{d}{d\epsilon} \left. I(u + \epsilon h) \right|_{\epsilon = 0} = \frac{1}{D(u)} \left. \frac{d}{d\epsilon} I(u + \epsilon h) \right|_{\epsilon = 0} - \frac{1}{Re_k} \left. \frac{d}{d\epsilon} D(u + \epsilon h) \right|_{\epsilon = 0} = 0.$$

Now differentiate the $I$ and $D$ terms:

$$\frac{d}{d\epsilon} I(u + \epsilon h) \left|_{\epsilon = 0} = -\frac{d}{d\epsilon} \langle u_j, h_j \rangle \right|_{\epsilon = 0}$$

$$= -\langle u_j h_j, F_j \rangle \left|_{\epsilon = 0} = -2(h_j, F_j) \right.$$ such that $F_y$ is symmetric

$$\frac{d}{d\epsilon} D(u + \epsilon h) \left|_{\epsilon = 0} = \frac{d}{d\epsilon} \left( \| \nabla (u + \epsilon h) \|^2 - \| \Delta (u + \epsilon h) \|^2 \right) \right|_{\epsilon = 0} = 2(\Delta u - MA^2 u, h_j).$$

The condition $h_{i,j} = 0$ can be included through the Lagrange multiplier $2\zeta(x)$, so that:

$$2(h_{i,j} \zeta) = -2(h_{i,j}).$$
Collecting together all terms in $h_i$, $\frac{I}{D}$ is an extremum provided that:

$$\langle (\Delta u_i - M\Delta^i u_i - Re\Delta u_i F_i + \zeta)h_i \rangle = 0.$$ 

As $h$ was selected as an arbitrary function, and if we identify $\zeta(x)$ with the pressure $\pi$, the Euler-Lagrange equations have the following form:

$$\Delta u_i - M\Delta^i u_i - Re\Delta u_i F_i = -\pi_i,$$

$$u_i = 0,$$ 

(15)

therefore, our aim is to solve (15) which represents an eigenvalue problem in $Re\Delta$ to find the smallest Reynolds number $Re\Delta$. By substituting $z_i = (U(z), 0, 0)$, in (15), the above system has the following final form:

$$\Delta u_i - M\Delta^i u_i - \frac{1}{2} Re U' = -\pi_i,$$

$$\Delta v_i = 0,$$

$$\Delta w_i - M\Delta^i w_i - \frac{1}{2} Re U' = -\pi_i,$$

$$u_i + v_i + w_i = 0.$$ 

(16)

The Euler-Lagrange equations which are resulted in (16) are not in the form such that Squire’s theorem can be applied. Therefore, here we examine the problems of $y-$ and $x-$ independent solutions.

4.1 Stability with respect to a $y-$independent perturbation

Let $u = u(x, z; t)$, then (15), reduces to:

$$\Delta u - M\Delta^i u - \frac{1}{2} Re U' = -\pi,$$

$$\Delta v = 0,$$

$$\Delta w - M\Delta^i w - \frac{1}{2} Re U' = -\pi,$$

$$u + v + w = 0,$$ 

(17)

where $\Delta = \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial z}$. It is clear that the $v$ equation uncouples and the solution is $v = 0$. Thus, we can remove $\pi$ by performing $\frac{\partial}{\partial z} \times (17_i) - \frac{\partial}{\partial x} \times (17_i)$, to have:

$$\Delta(u_i - w_i) - M\Delta^i (u_i - w_i) + \frac{1}{2} Re (u_i - w_i)U' - \frac{1}{2} Re U'' = 0,$$

Differentiating above equation with respect to $x$ and using (17), to substituting $u_i = w_i$ and $u_i = w_i$, we arrive at the eigenvalue problem:

$$\Delta^i w - M\Delta^i w + Re U'' U' + \frac{1}{2} Re U'' = 0,$$ 

(18)

where $z \in (-1, 1)$, we consider the solutions of equation (18) as the form $w = w(z)e^{ix}$ and substituting this form into (18), we obtain:

$$(D^2 - a^2)^2 w - M(D^2 - a^2)^2 w + i a Re(U'Dw + \frac{1}{2} wU'') = 0.$$ 

(19)

where $w$ satisfies (12).

4.2 Stability with respect to a $x-$independent perturbation

Now, for stability with respect to $x-$ independent perturbation, we suppose that $u = u(y, z; t)$, then (15) reduces to:
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\[
\begin{align*}
\Delta u - M \Delta^2 u - \frac{1}{2} \text{Re} u' &= 0, \\
\Delta v - M \Delta^2 v &= -\pi, \\
\Delta w - M \Delta^2 w - \frac{1}{2} \text{Re} u' &= -\pi, \\
v_y + w_y &= 0,
\end{align*}
\]

we can eliminate \( \pi \) by performing \( \frac{\partial}{\partial y} \times (20)_2 - \frac{\partial}{\partial y} \times (20)_3 \), yields:

\[
\Delta (v_y - w_y) - M \Delta^2 (v_y - w_y) + \frac{1}{2} \text{Re} u' U' = 0,
\]

differentiating above equation with respect to \( y \) and using \( (20)_4 \) to substituting \( v_{yy} = -w_{yy} \) and \( v_{zy} = -w_{zy} \), yields:

\[
\Delta^2 w - M \Delta^2 w - \frac{1}{2} \text{Re} U' U' = 0.
\]

Now consider the solutions of equation (21) as the form \( u = u(z)e^{iy} \) and \( w = w(z)e^{iy} \) and substituting this form into (21) and (20):

\[
\begin{align*}
2(D^2 - a^2)^2 w - 2M(D^2 - a^2)^2 w + a^2 \text{Re} u' &= 0, \\
2(D^2 - a^2)u - 2M(D^2 - a^2)^2 u - \text{Re} U' &= 0
\end{align*}
\]

5. Numerical Techniques

Since solving equation (11) is a very difficult numerical problem for finding the eigenvalues, we will utilize a very accurate numerical method, which is the Chebyshev collocation method. The eigenvalue systems of linear instability and nonlinear stability theories have been solved using the Chebyshev collocation method, for more details see [25-33]. As these texts point out that the advantage in using the Chebyshev collection method is that it can achieve the required accuracy using a small number of polynomials, allowing the achievement of highly accurate results with short run time. Moreover, this method has the highest accuracy between the numerical methods and requires a smaller number of polynomials to achieve excellent accuracy and convergence. Also, the above texts note that the Chebyshev collection method is one of the best choices in solving the hydrodynamic stability problems as it is a flexible method and it can give very accurate results.

Firstly, the functions \( A = Dw \) and \( B = D^3 w \) are introduced, then equation (11) can be written in the following form:

\[
\begin{align*}
Dw - A &= 0, \\
D^2 A - B &= 0, \\
(a^2 + a'M + ia \text{Re} U' + ia \text{Re} U')w - (2a^2 + 3a'M + ia \text{Re} U')DA \\
+ (D - MD^3 + 3a^2 MD)B &= ia'e^{iy} - iac \text{Re} DA.
\end{align*}
\]

The boundary conditions (12) have the form:

\[
A = B = w = 0, \quad z = \pm 1
\]

To use the Chebyshev collocation method, we write solutions for the equations (23) and (24) as the sum of a limited number of Chebyshev polynomials, so these solutions take the following forms:

\[
\begin{align*}
w &= \sum_{n=0}^{N} w_n T_n(z), \quad A = \sum_{n=0}^{N} A_n T_n(z), \quad B = \sum_{n=0}^{N} B_n T_n(z).
\end{align*}
\]

where \( T_n(z) \) are the Chebyshev polynomials of the first kind, see [3], which is defined by:

\[
\begin{align*}
T_0(z) &= 1, \quad T_1(z) = z \\
T_{n+1}(z) - 2z T_n(z) + T_{n-1}(z) &= 0, \quad -1 \leq z \leq 1
\end{align*}
\]

or

\[
T_n(z) = \cos(n \arccos(z)), \quad -1 \leq z \leq 1.
\]
The expression (25) is employed in equations (23) and (24) and then the resulting equations are computed at Gauss-Lobatto points $y_i$ which are defined by $y_i = \cos(\pi i / [N - 2])$, $i = 0, ..., N - 2$. This gives $3N - 3$ equations for $3N + 3$ unknowns $w_0, ..., w_N, A_0, ..., A_N, B_0, ..., B_N$. The remaining 6 equations are furnished by the boundary conditions (24) which becomes:

$$
BC_1 : \sum_{n=0}^{N} w_n = 0, \quad BC_2 : \sum_{n=0}^{N} (-1)^n w_n = 0, \quad BC_3 : \sum_{n=0}^{N} A_n = 0,$$

$$
BC_4 : \sum_{n=0}^{N} (-1)^n A_n = 0, \quad BC_5 : \sum_{n=0}^{N} B_n = 0, \quad BC_6 : \sum_{n=0}^{N} (-1)^n B_n = 0, \quad (26)
$$

Then approximated boundary conditions (26) are added as rows to the generated matrices to yield a $(3N + 3) \times (3N + 3)$ eigenvalue matrix. Thus, we have the eigenvalue system:

$$
X = c \begin{pmatrix} O & O & O \\ O & O & O \\ O & O & O \\ \vdots & \vdots & \vdots \\ O & O & O \\ O & O & O \\ O & O & O \\ O & O & O \\ \end{pmatrix} X,
$$

where $X = (w_0, ..., w_N, A_0, ..., A_N, B_0, ..., B_N)$, $O$ is the zeros matrix,

$$
A_1(n_1, n_2) = (a^4 + a^5 M + ia ReU'''(y_{n_1}) + ia^2 ReU'(y_{n_1})) I(n_1, n_2),
$$

$$
A_2(n_1, n_2) = -(2a^2 + 3a^4 M + ia ReU'(y_{n_1})) D(n_1, n_2),
$$

$$
D(n_1, n_2) = (1 + 3a^2 M) D(n_1, n_2) - MD^3(n_1, n_2)
$$

$$
I(n_1, n_2) = T_{2_2}^{(n_1)}(y_{n_1}), \quad D(n_1, n_2) = T_{2_2}^{(n_1)}(y_{n_1}), \quad D^2(n_1, n_2) = T_{2_2}^{(n_1)}(y_{n_1}), \quad D^3(n_1, n_2) = T_{2_2}^{(n_1)}(y_{n_1}),
$$

$$
n_1 = 0, ..., N - 2, \quad n_2 = 0, ..., N.
$$

The above system has been solved using the QZ algorithm. For a perturbation $u, v, w, \pi$ dependent on $x, z$, we implement the Chebyshev collocation method to the eigenvalue problem (29) to obtain the linear system:

$$
(D^2 - a^2)^2 w - M (D^2 - a^2)^2 w + ia Re(U'Dw + \frac{1}{2} wU') = 0. \quad (29)
$$

where $X = (w_0, ..., w_N, A_0, ..., A_N, B_0, ..., B_N)$, $A_1(n_1, n_2) = \frac{1}{2} ia U'''(y_{n_1}) I(n_1, n_2)$, $A_2(n_1, n_2) = ia U'''(y_{n_1}) I(n_1, n_2)$, and $n_1 = 0, ..., N - 2, \quad n_2 = 0, ..., N$. However, for the eigenvalue system for $x$ — independent solutions (22), the Chebyshev collocation method yields the eigenvalue system of the form:
\[ \hat{AX} = \text{Re}\hat{B}, \]  
\[ (30) \]

where

\[ \hat{A} = \begin{pmatrix} D & -I & 0 & 0 & 0 \\ BC_1 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ BC_2 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ O & D^2 & -I & 0 & 0 \\ O & 0, \ldots, 0 & BC_3 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ O & 0, \ldots, 0 & BC_4 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ A_1 & A_2 & A_3 & 0 & 0 \\ O & O & O & D^2 & -I \\ O & 0, \ldots, 0 & 0, \ldots, 0 & BC_5 & 0, \ldots, 0 & 0, \ldots, 0 \\ O & 0, \ldots, 0 & 0, \ldots, 0 & BC_6 & 0, \ldots, 0 & 0, \ldots, 0 \\ O & O & O & A_6 & A_7 \\ O & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & BC_8 & 0, \ldots, 0 \\ O & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & BC_9 \\ O & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & BC_{10} \end{pmatrix} \]

\[ (31) \]

\[ \hat{B} = \begin{pmatrix} O & O & O & O & O \\ 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ O & O & O & O & O \\ 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ O & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ O & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ A_8 & 0 & 0 & O & O \\ 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \\ 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 & 0, \ldots, 0 \end{pmatrix} \]

\[ (32) \]

\[ X = (w_0, \ldots, w_N, A_0, \ldots, A_N, B_0, \ldots, B_N, u_0, \ldots, u_N, C_0, \ldots, C_N), \]

\[ C = D^2u, \quad BC_7 : \sum_{n=0}^{N} u_n = 0, \quad BC_8 : \sum_{n=0}^{N} (-1)^n u_n = 0, \quad BC_9 : \sum_{n=0}^{N} C_n = 0, \quad BC_{10} : \sum_{n=0}^{N} (-1)^n C_n = 0, \]

\[ A_4(n_1, n_2) = -(a^2 + Ma^2)I(n_1, n_2), \quad A_4(n_1, n_2) = (1 + 2a^2M)I(n_1, n_2) - MD^2(n_1, n_2), \]

\[ A_4(n_1, n_2) = -\frac{1}{2} a^2U'(y \cdot x)I(n_1, n_2), \quad A_4(n_1, n_2) = \frac{1}{2} U'(y \cdot x)I(n_1, n_2), \]

\[ (33) \]

\[ (34) \]

For a given \( a \) and \( M \), then we use QZ algorithm of Matlab routines to solve the eigenvalue system, and next, we iterate this procedure over choices of \( a \), tracking the smallest eigenvalue until we have found the critical Reynolds number which we denote by \( Re_c \).
6. Numerical Results

In Table 1, the numerical results for the critical value $Re$ are displayed for linear instability and nonlinear energy stability theory. It is very clear that there is a big difference between linear and nonlinear thresholds. This difference shows that there is a potential region of subcritical instabilities area may arise.

The critical Reynolds numbers are shown in Fig. 3, where instability curves were plotted as a function of the couple stresses coefficient $M$. The curves can be explained as follows. For instance, when $0 < M \leq 0.002$, the instability curve has been shown in Fig. 3 (a). For $Re$ values below this curve, the solution is linearly stable, and the imaginary part of all eigenvalues is negative, i.e. $c < 0$. For $Re$ values that are above this curve, then we should have at least one eigenvalue which has the imaginary part is positive, i.e. $c > 0$ and the solution grows exponentially and is unstable. A similar explanation holds for the other Figures. We note that as $M$ increases, the critical Reynolds number increases substantially. This shows the strong stabilizing effect of the couple stresses coefficient $M$. Thus, the increasing $M$ value leads to strongly increase the threshold at which instability commences.

The critical values of $r_c$ as a function of $M$ have shown in Fig. 4. As $M$ vary, these values refer to how oscillatory the solution is in time at the start of instability.

The critical wavenumber curves against $M$ are displayed in Fig. 5, where $M$ varying over the range $M=0$ to $M=1$. These curves are explained as those for $Re_L$, where above these curve we have instability region, while the region below these curves are linearly stable. We also note that the values of critical wavenumber increase with increasing the values $M$ and this means that the periodic cells of the $w$ become smaller in the $x$ and $y$ directions.

In Fig. 6, the eigenvalues spectrum are introduced and we have found that are similar to that found for Poiseuille flow in a porous medium with no-slip boundary conditions in [21] and slip boundary conditions in [9]. The eigenvalues displaying a $Y$ shape in the $(c_v,c_f)$ diagram. As $M$ increase, the eigenvalues at the intersection of the three lines in the $Y$ become more numerically unstable, and this effect becomes clearer when the value of $M$ is increased.
Poiseuille flow with couple stresses effect and no-slip boundary conditions

The spectrum of (23) and (24) acting very like to that of the Orr-Sommerfeld problem for classical Poiseuille flow. For example, for higher Reynolds numbers we witnessed mode crossing of eigenvalues. For instance, for $M = 0.5$, the first and second eigenvalues interchange places for $Re$ between 55500 and 55502, respectively, with the previous first eigenvalue moving down the list as $Re$ increases. This behavior is very like to that noted in [3]. Also, the spectrum is highly sensitive and the number of polynomials that are used in the numerical approximation should be carefully selected, and in the arithmetical precision used in the calculation (those shown here are all 64 bit arithmetic). The spectrum for $M = 0.3$, $a = 5$ and $Re = 10^6$ are shown in Fig. 5, respectively, for different values of $N$. Since $U$ is an even function of $z$, then, the proper solution of the eigenvalue system (23) and (24) locates in two unincorporated groups for even and odd solutions.

An interesting question for this problem can appear which relates to the accuracy of the expected results when comparing the results of linear instability and nonlinear stability theories. Therefore, this problem is one of the most difficult flow problems even in a clear fluid with slip and no-slip boundary conditions, see e.g. [2, 9]. The results of the linear instability guarantee that the solution is not stable for the Rayleigh number which exceeds the linear threshold. However, it does not guarantee stability when the Rayleigh number is below this. The nonlinear energy stability thresholds of nonlinear show the stability only where the solution is stable if the Rayleigh number is below this threshold, however, they say nothing about instability. Therefore, in this paper, we compute the nonlinear energy stability thresholds for our problem, just as in porous medium with the conditions of no-slip boundary by Hill & Straughan [11], and as was done for a clear fluid with slip boundary conditions by Webber & Straughan [8]; the work which are also carefully reported in chapter 3 of Webber [22]. For the current problem, the nonlinear energy stability thresholds which confirm the area of the stability of the solution are well below those of linear theory. This is one field in which the theory of nonlinear stability is not useful, and from these not unknown scenarios in other fields of mechanically fluid, see. Straughan [23], where there is a big difference discrepancy between the linear instability and the nonlinear stability thresholds. This shows that there is a potential region of subcritical instabilities area may arise. A fully three-dimensional simulation for some problems [24-29] shows that there is a region of subcritical instability below the linear instability threshold, but well above the nonlinear energy stability boundary.

*Fig. 4. Critical value of $c_r$ against $M$.\*
Fig. 5. Critical wavenumber $a_L$ against $M$.

Table 1. Critical Reynolds number $Re$, wavenumber $a$ and wave speed $c_r$ against couple stresses number $M$.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$Re_{cr}$</th>
<th>$a_L$</th>
<th>$c_r$</th>
<th>$Re^{a}_{cr}$</th>
<th>$a_{cr}^{a}$</th>
<th>$Re^{c}_{cr}$</th>
<th>$a_{cr}^{c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>9151.467</td>
<td>1.556</td>
<td>0.38</td>
<td>365.39</td>
<td>1.862</td>
<td>140.967</td>
<td>1.72</td>
</tr>
<tr>
<td>0.2</td>
<td>17150.775</td>
<td>1.58</td>
<td>0.325</td>
<td>720.667</td>
<td>1.827</td>
<td>259.884</td>
<td>1.661</td>
</tr>
<tr>
<td>0.3</td>
<td>27174.461</td>
<td>1.589</td>
<td>0.282</td>
<td>1167.031</td>
<td>1.814</td>
<td>408.652</td>
<td>1.634</td>
</tr>
<tr>
<td>0.4</td>
<td>39232.453</td>
<td>1.595</td>
<td>0.248</td>
<td>1705.333</td>
<td>1.806</td>
<td>587.354</td>
<td>1.62</td>
</tr>
<tr>
<td>0.5</td>
<td>53325.548</td>
<td>1.598</td>
<td>0.221</td>
<td>2335.794</td>
<td>1.796</td>
<td>796.01</td>
<td>1.61</td>
</tr>
<tr>
<td>0.6</td>
<td>69454.032</td>
<td>1.6</td>
<td>0.2</td>
<td>3058.484</td>
<td>1.797</td>
<td>1034.628</td>
<td>1.603</td>
</tr>
<tr>
<td>0.7</td>
<td>87617.993</td>
<td>1.602</td>
<td>0.182</td>
<td>3873.448</td>
<td>1.79</td>
<td>1303.209</td>
<td>1.599</td>
</tr>
<tr>
<td>0.8</td>
<td>107817.56</td>
<td>1.602</td>
<td>0.167</td>
<td>4780.716</td>
<td>1.788</td>
<td>1601.757</td>
<td>1.595</td>
</tr>
<tr>
<td>0.9</td>
<td>130052.74</td>
<td>1.603</td>
<td>0.155</td>
<td>5780.265</td>
<td>1.787</td>
<td>1930.271</td>
<td>1.592</td>
</tr>
<tr>
<td>1</td>
<td>154323.72</td>
<td>1.604</td>
<td>0.144</td>
<td>6872.148</td>
<td>1.786</td>
<td>2288.752</td>
<td>1.59</td>
</tr>
</tbody>
</table>

Table 2. Comparison between the critical Reynolds numbers (minimized over the wavenumber) of Takashima [20], Hill & Straughan [11], Harfash [34], and this paper for $M = 0$.

<table>
<thead>
<tr>
<th>Methods</th>
<th>$R_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Takashima [20], Chebyshev collocation by imposing BCs.</td>
<td>5772.2218</td>
</tr>
<tr>
<td>Harfash [34] Chebyshev collocation method with even Polynomials</td>
<td>5772.22198</td>
</tr>
<tr>
<td>Harfash [34], Finite element</td>
<td>5771.920022</td>
</tr>
<tr>
<td>The present work</td>
<td>5772.222169</td>
</tr>
</tbody>
</table>

Finally, in order to validate our numerical method, we compare the critical Reynolds numbers of Takashima [20], Hill & Straughan [11], Harfash [34], and this paper for $M = 0$. In the other studies $M$ has different definitions but $M = 0$ the equivalent of the Orr-Sommerfeld eigenvalue problem becomes identical and therefore the critical Reynolds numbers should also identical. It is noted that there is very little variation with the critical Reynolds numbers observed by the other studies, as is shown in Table 2. This confirms the high accuracy of the numerical method which is used in this paper.

Fig. 6. Spectral of growth rate $c = c_r + ic_i$ at $M = 0.3$. 

(a) $N = 75$

(b) $N = 100$
7. Conclusions

A linear instability analysis and nonlinear stability analyses for Poiseuille under the effect of couple stresses were presented. In deriving the equations governing the stability, a simplification was made using the fact that and the flow was driven by a constant pressure gradient in the direction. Using modified Squire’s transformations, it was established the nonlinear stability for and disturbances. Also, the Chebyshev collocation method with the algorithm was used for solving the stability equations to find the eigenvalues. The secant and the golden section search were also utilized to compute the critical values. The critical numerical values of Reynolds number, wave number, and wave speed were computed for several selected values of the couple stress coefficient. For Poiseuille flow, we can conclude that the couple stresses have a stabilizing effect on the flow where as increases, the critical Reynolds number increases to stabilize the flow. The results of the present study confirm the impact of the non-Newtonian effects on the flow instability. Hence, further works can be provided to study the linear instability and nonlinear stability behaviors of new fluids such as suspension and slurries of nano-encapsulated phase change materials, see for examples [35-39].

Author Contributions

The authors conceived the mathematical model, proved the mathematical results and wrote the paper together. The authors gave their final approval for publication.

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Conflict of Interest

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