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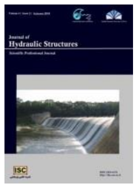

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Boundary Integral Equations for Quasi-Static Unsaturated Porous Media

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Abstract

One of the principal criteria for development of the boundary element method (BEM) in porous media is derivation of the required fundamental solutions in the boundary integral equations (BIE). Furthermore, setting up the governing BIEs based on the governing partial differential equations (PDE) is another challenge in solving a physical phenomenon using BEM. In this regard, the governing BIEs for unsaturated porous media have been developed using the available derived fundamental solutions. In this research, a perturbation type approximation is exploited for developing a system of BIEs for the quasi-static unsaturated porous media with moderate variations in its properties. Nevertheless, the fundamental solutions of the medium with constant properties are applied. The method produces two sets of equations with constant parameters instead of the original equations. Besides, the required boundary conditions have been formulated. This type of BIEs is essential to be used in the BEM for unsaturated porous media as the fundamental solutions for a medium with coordinates dependent properties is not available so far. The resulted introduced BIEs may be used directly in a BEM numerical model for an unsaturated porous media in one, two or three dimensional conditions.

Keywords: Unsaturated porous media, Boundary Integral Equation, Fundamental Solutions, Quasi-Static, Perturbation.

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1. Introduction

Unsaturated soil, as a prevailing medium that surrounds most of the structures, has been of great interest during four past decades. Therefore, a considerable number of researches have been devoted to modeling its characteristics. The Biot's theory is often used as a mathematical model for the dynamic behavior in saturated soils. Its application to unsaturated soil problems is

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possible under certain sets of conditions [1]. The Biot's theory has been extended to nearly saturated porous media by Aifantis and Wilson and Aifantis for the quasi-static case, however, there are several other theories for partially saturated media or media with more phases [2].

BIE methods are among the most efficient numerical methods which depends strongly on finding the fundamental solutions of the governing homogenous PDEs, especially in the BEM. Therefore, many efforts have been devoted on finding the fundamental solutions for different cases of the problem. The first set of fundamental solutions for saturated media has been introduced by Cleary [3]. Then, several researches have been published on the fundamental solutions for different phenomena of saturated porous media such as deformation and heat conduction. In contrast, the fundamental solutions for unsaturated media have been published recently. Gatmiri and Jabbari derived the first static and quasi-static fundamental solutions of the problem in both Laplace and time domains [4], [5]. Later, Maghoul et al. presented coupled thermo-hydro-mechanical fundamental solutions for the same quasi-static loading condition of the unsaturated soil for two and three-dimensional time domains [6]. Ashayeri et al. introduced fundamental solutions for the dynamic problem in both 2D and 3D cases. A similar problem has been studied by Li and Schanz [7]. Ghorbani et al [8] studied the non-linear behavior of the solid skeleton of the soil in the analysis of multiphase unsaturated soils when subjected to both static and dynamic loading. Igumnov et al [9] considered wave propagation in fully and partially saturated porous media with examples of two-components and three-components. Igumnov et al [10] deduced the solution of a finite one-dimensional column with Neumann and Dirichlet boundary conditions based on the theory of mixture.

More capabilities of the BEM may be realized when the required fundamental solutions to be available. Unfortunately, it is not always the case, instantly, for inhomogeneous soils, where the fundamental solutions have been derived only in a few very particular cases e.g. when the Poisson's ratio is equal to 0.25 [11]. In the absence of the required fundamental solution the bothering problems of the BEM such as domain integrals on the domain will arise.

In unsaturated porous media, the state variables are the stress tensor, air and water pressures that change spatially, so a nonhomogeneous medium must be considered. In such a case, the boundary elements method seems to be ineffective. In detail, for the quasi-static case of the porous media the fundamental solutions in hand, which have been found for a homogeneous medium, are not efficient. Also, the assumptions that have been considered for finding the fundamental solutions for other media do not appear useful here e.g. when the Poisson's ratio is constant or when the variation of the state variables is a predefined pattern. Hence, in this paper, two boundary equation sets have been developed for the problem, when the parameters of the medium are varied slightly near a mean value. The first set concerned with a medium that has constant parameters and the second considers the effects of variation of the parameters. The toll was the appearance of a secondary set of equations which should be resolved. These sets of equations must be applied to make a BEM model. Additionally, more accuracy may be desired when large variations in the medium handled by employing the multi-region technique.

In this paper the governing equations are briefly reviewed. Then changes in different parameters of the medium are found in terms of the changes of the state parameters and the governing equations are rewritten using this perturbed type of parameters. Then the integral equations have been developed and the boundary conditions are revised for the effects of the changes in state parameters. Finally, two sets of boundary integral equations have been constituted up which employ the fundamental solutions of the homogeneous medium.

2. Governing Equations

The effective stress of the matrix of a porous medium may be written in terms of the displacements and pressures of fluid. For an elastic isotropic medium, the constitutive equations are [12]:

$$\left(\sigma_{ij}+\delta_{ij}P^{a}\right)=\lambda\delta_{ij}u_{k,k}+\mu\left(u_{i,j}+u_{j,i}\right)+\delta_{ij}D_{s}\left(P^{a}-P^{w}\right) \quad (1)$$

Where λ and μ are the Lame's coefficients for the skeleton, P^a and P^w are the gas (air) and the liquid (water) phase pressures, respectively, and D_s is the isotropic Biot's coefficient for the fluid phase [12]. For a nonlinear elastic material, the equations may be linearized by the incremental form as [4]:

$$d\left(\sigma_{ij}+\delta_{ij}P^{a}\right)=D_{ijkl}d\varepsilon_{kl}+\delta_{ij}D_{s}\left(dP^{a}-dP^{w}\right) \quad (2)$$

The parameters in eq. (1) are functions of the state variables, but in the incremental form (eq. (2)) they are functions of the spatial coordinates.

The momentum balance equation for the medium with the constitutive equation (2) and ignoring accelerations (to model the quasi-static case) of distinct phases could be written as [4]:

$$\left(\lambda+\mu\right)du_{j,i}+\mu du_{i,j}-\delta_{ij}\left(1-D_{s}\right)dP^{a}-\delta_{ij}D_{s}dP^{w}=0 \quad (3)$$

The mass of medium does not play any role in the incremental equation because it does not change, besides, the acceleration has been dropped. Furthermore, we can replace the saturation ratio for D_s to find a simpler equation [13].

The saturation ratio, like every other parameter of the medium, is governed by the state variables. But we use a simple one which only needs for the suction [14] (a complete form could be seen in [9]):

$$Sr=\alpha+\beta Log\left(P^{a}-P^{w}\right) \quad (4)$$

In this equation, Sr stands for the saturation ratio, α and β are constants.

For investigation of the mass balance of distinct phases a moving control volume attached to the solid skeleton has been considered that ensures the mass balance of the solid phase, but balance of other phases should be certified. Assumption of the incompressibility of the liquid phases let replacing the balance of volume equations for the balance of masses:

$$\frac{\partial}{\partial t}\left(n\left(1-Sr\right)\right)+q_{i,i}^{a}=0 \quad (5)$$

$$\frac{\partial}{\partial t}\left(nSr\right)+q_{i,i}^{w}=0 \quad (6)$$

Where q_i^α denotes the fluid volume fluxes $\left(\alpha=a\, or\, w\right)$. It should be noted that the solution of air in the water has been ignored here, which may restrict application of these equations. However, in the case of constant ratio of dissolved air, they could be said by getting the mixture of water and dissolved air as the liquid phase. Equation (4) is needed in the incremental form which reveals the increment of the saturation ratio in terms of the increments of the state variables:

$$dSr=\beta\frac{\left(dP^{a}-dP^{w}\right)}{(\hat{P}^{a}-\hat{P}^{w})} \quad (7)$$

Where $\hat{P}^a$ and $\hat{P}^w$ stand for the pressures of air and water in the last step, respectively.

Darcy's law has been exploited to evaluate the fluxes and then the continuity equation for gas and liquid phases could be written as [4]:

$$-\left(\frac{K^w}{\gamma^w} dP^w_{,i}\right)_{,i} + Srd\dot{u}_{i,i} + \frac{\beta\hat{n}}{(\hat{P}^a - \hat{P}^w)}(d\dot{P}^a - d\dot{P}^w) = 0 \quad (8)$$

$$-\left(\frac{K^w}{\gamma^w} dP^w_{,i}\right)_{,i} + Srd\dot{u}_{i,i} + \frac{\beta\hat{n}}{(\hat{P}^a - \hat{P}^w)}(d\dot{P}^a - d\dot{P}^w) = 0 \quad (9)$$

Where the dot stands for a temporal derivation. Equations (3), (8) and (9) form the governing equations of the problem.

The Laplace transform is a perfect and usual tool for solving such a problem [12], [4], [3]. The following equations are achieved after applying the Laplace transform on the governing equations and assuming the parameters of the medium to be constants:

$$c_{11} d\tilde{u}_{j,ij} + c_{12} d\tilde{u}_{i,jj} + c_{13} d\tilde{P}^a_{,i} + c_{14} d\tilde{P}^w_{,i} = 0 \quad (10)$$

$$c_{21} d\tilde{u}_{i,i} + c_{22} d\tilde{P}^a + c_{23} d\tilde{P}^a_{,ii} + c_{24} d\tilde{P}^w = 0 \quad (11)$$

$$c_{31} d\tilde{u}_{i,i} + c_{32} d\tilde{P}^a + c_{33} d\tilde{P}^w_{,ii} + c3 d\tilde{P}^w = 0 \quad (12)$$

In which the tiled mark (~) stands for the Laplace domain variables. In addition, for an incremental model, the initial conditions are assumed to be zero. The c_{ij} coefficients are:

$$\begin{aligned}
c_{11} &= (\lambda + \mu) \\
c_{12} &= \mu \\
c_{13} &= -(1 - Sr) \\
c_{14} &= -Sr \\
c_{21} &= s(1 - Sr) \\
c_{22} &= -c_{24} = -c_{32} = c_{34} = \frac{-s\beta\hat{n}}{(\hat{P}^a - \hat{P}^w)} \\
c_{23} &= -\frac{K^a}{\gamma^a} \\
c_{31} &= sSr \\
c_{33} &= -\frac{K^w}{\gamma^w}
\end{aligned} \quad (13)$$

The spatial variations of the state variables cause the parameters of the media to experience some spatial variations. Nevertheless, only the fundamental solutions are developed for constant parameters and are not applicable directly here. For using these fundamental solutions, it has been assumed that the state variables change slightly in the domain:

$$\frac{\sigma'_{ij}(x)}{P^c(0)} = \frac{\sigma'_{ij}(0)}{P^c(0)} + \varepsilon\theta_{\sigma_{ij}}(x) \quad (14)$$

$$\frac{P^a(x)}{P^c(0)} = \frac{P^a(0)}{P^c(0)} + \varepsilon\theta_a(x) \quad (15)$$

$$\frac{P^w(x)}{P^c(0)}=\frac{P^w(0)}{P^c(0)}+\varepsilon\theta_w(x) \tag{16}$$

In these equations ε is a constant in some trivial size and $\theta_\alpha(x)\ ;\alpha=\sigma'_{ij},a,w$ are functions of the coordinates which show the distribution of the variables. These functions are of order one ($O(1)$) and cannot violate the assumption of slight changes in the state variables. The effective stress field is shown by $\sigma'(x)$ and the pressure fields of fluids are denoted by $P^a(x)$ and $P^w(x)$ for air and water, respectively. In equations (14) to (16) the fields of the state parameters have been revealed in terms of their value in the origin and the ratio of their variation to the value of the suction at the origin. The origin should be chosen properly to satisfy these equations and the domain could be divided to distinct regions, if necessary.

The suction pressure could be considered as a function of the saturation ratio [14] and its value for some types of soils could be very large. Therefore, the relative changes of the suction could be small (see eq. (7)). Additionally, the resistance to the air flow is very small that the air pressure could be assumed equal to the atmosphere pressure at all points of the system or, at least, vary slightly [13]. Consequently, equations (15) and (16) could be claimed. The ratio of the variations of the effective stress and the suction pressure has been assumed small, so, equation (14) could be written.

The variations in the parameters affect the solution which could be assessed using multi variable Taylor's series. Here, only the first orders have been considered and the results are:

$$\begin{aligned} du_i\left(x,\frac{\sigma'_{ij}(x)}{P^c(0)},\frac{P^a(x)}{P^c(0)},\frac{P^w(x)}{P^c(0)}\right) &= du_i\left(x,\frac{\sigma'_{ij}(0)}{P^c(0)},\frac{P^a(0)}{P^c(0)},\frac{P^w(0)}{P^c(0)}\right) \\ &+\varepsilon\left(\frac{\partial du_i}{\partial(\sigma'_{ij}/P^c(0))}\theta_{\sigma'_{ij}}+\frac{\partial du_i}{\partial(P^a/P^c(0))}\theta_{P^a}+\frac{\partial du_i}{\partial(P^w/P^c(0))}\theta_{P^w}\right) \\ &= du0_i(x)+\varepsilon du1_i(x) \end{aligned} \tag{17}$$

$$dP^a\left(x,\frac{\sigma'_{ij}(x)}{P^c(0)},\frac{P^a(x)}{P^c(0)},\frac{P^w(x)}{P^c(0)}\right)=dP0^a(x)+\varepsilon dP1^a(x) \tag{18}$$

$$dP^w\left(x,\frac{\sigma'_{ij}(x)}{P^c(0)},\frac{P^a(x)}{P^c(0)},\frac{P^w(x)}{P^c(0)}\right)=dP0^w(x)+\varepsilon dP1^w(x) \tag{19}$$

There is a perturbation type expansion, which approximates the solutions of the equations as a combination of solutions for the medium with constant parameters including a small perturbation. This result could be a premium assumption in other perturbation method equations such as in [15] and [16].

In appendix A, it has been shown that all other parameters could be written in the form of equation (20). Namely, for a typical parameter F:

$$F\left(x,\frac{\sigma'_{ij}(x)}{P^c(0)},\frac{P^a(x)}{P^c(0)},\frac{P^w(x)}{P^c(0)}\right)=F(0)(1+\varepsilon\theta_F(x)) \tag{20}$$

Where $F(0)$ is the value of the typical parameter F in the origin and θ_F is a normalized function of coordinates to account the spatial changes of factor F.
Using the coordinate dependent parameters in the governing equations, computing the derivatives and then ignoring the second and higher orders of ε lead to a new set of equations based on ε and the theta functions. These equations could be separated into two sets, one consists of the terms without ε (or with ε^0) and the other has the terms with ε (or with ε^1) named zero and first order equations, respectively. The zero order equations are related to an imagined medium constant parameters which could be found at the origin and the other set reflects the effects of derivatives of the parameters when ε has not been vanished. They are as follow:
The zero order equations:

$$(\lambda(0)+\mu(0))d\tilde{u}0_{j,ij}+\mu(0)d\tilde{u}0_{i,jj}-(1-Sr(0))d\tilde{P}^a0_{,i}-Sr(0)d\tilde{P}^w0_{,i}=0 \tag{21}$$

$$-K^a(0)d\tilde{P}0^a{}_{,ii}+s(1-Sr(0))d\tilde{u}0_{i,i}-s\frac{\beta n(0)}{P^c(0)}\left(d\tilde{P}0^a-d\tilde{P}0^w\right)=0 \tag{22}$$

$$-K^w(0)d\tilde{P}0^w{}_{,ii}+sSr(0)d\tilde{u}0_{i,i}+s\frac{\beta n(0)}{P^c(0)}\left(d\tilde{P}0^a-d\tilde{P}0^w\right)=0 \tag{23}$$

In which P^c stands for the suction pressure P^a-P^w and $P^c(0)$ is its value in the origin and both are in the earlier time step.
The first order equations:

$$(\lambda(0)+\mu(0))d\tilde{u}1_{j,ij}+\mu(0)d\tilde{u}1_{i,jj}-(1-Sr(0))d\tilde{P}^a1_{,i}-Sr(0)d\tilde{P}^w1_{,i}=\tilde{F}_i \tag{24}$$

$$-K^a(0)d\tilde{P}1^a{}_{,ii}+s(1-Sr(0))d\tilde{u}1_{i,i}-s\frac{\beta n(0)}{P^c(0)}\left(d\tilde{P}1^a-d\tilde{P}1^w\right)=\tilde{F}^a \tag{25}$$

$$-K^w(0)d\tilde{P}1^w{}_{,ii}+sSr(0)d\tilde{u}1_{i,i}+s\frac{\beta n(0)}{P^c(0)}\left(d\tilde{P}1^a-d\tilde{P}1^w\right)=\tilde{F}^w \tag{26}$$

In which the right-hand side values are:

$$\begin{aligned}\tilde{F}_i&=-\lambda(0)\left(\theta_\lambda d\tilde{u}0_{i,jj}+\theta_{\lambda,i}d\tilde{u}0_{j,j}\right)-\\&\mu(0)\left(\theta_\mu\left(d\tilde{u}0_{i,jj}+d\tilde{u}0_{j,ij}\right)+\theta_{\mu,j}\left(d\tilde{u}0_{i,j}+d\tilde{u}0_{j,i}\right)\right)-Sr(0)\left(\theta_{Sr}dP0^c_{,i}+\theta_{Sr,i}dP0^c\right)\end{aligned} \tag{27}$$

$$\begin{aligned}\tilde{F}_a&=-sSr(0)\theta_{Sr}d\tilde{u}0_{i,i}+\\&K^a(0)\left(\theta_{K^a}d\tilde{P}0^a_{,ii}+\theta_{K^a,i}d\tilde{P}0^a_{,i}\right)+s\frac{\beta n(0)}{P^c(0)}\left(\theta_n-\theta_{P^c}\right)\left(d\tilde{P}0^a-d\tilde{P}0^w\right)\end{aligned} \tag{28}$$

$$\begin{aligned}\tilde{F}_w&=-sSr(0)\theta_{Sr}d\tilde{u}0_{i,i}+\\&K^w(0)\left(\theta_{K^w}d\tilde{P}0^w_{,ii}+\theta_{K^w,i}d\tilde{P}0^w_{,i}\right)-s\frac{\beta n(0)}{P^c(0)}\left(\theta_n-\theta_{P^c}\right)\left(d\tilde{P}0^a-d\tilde{P}0^w\right)\end{aligned} \tag{29}$$

Where the values that denoted with (0) (such as $\lambda(0)$) have been measured at the origin and at the earlier step. In addition, the zero order solution values (for example $du0_i$) are known values when the first order equations are going to be solved. The computational procedure has been illustrated in Fig. 1.

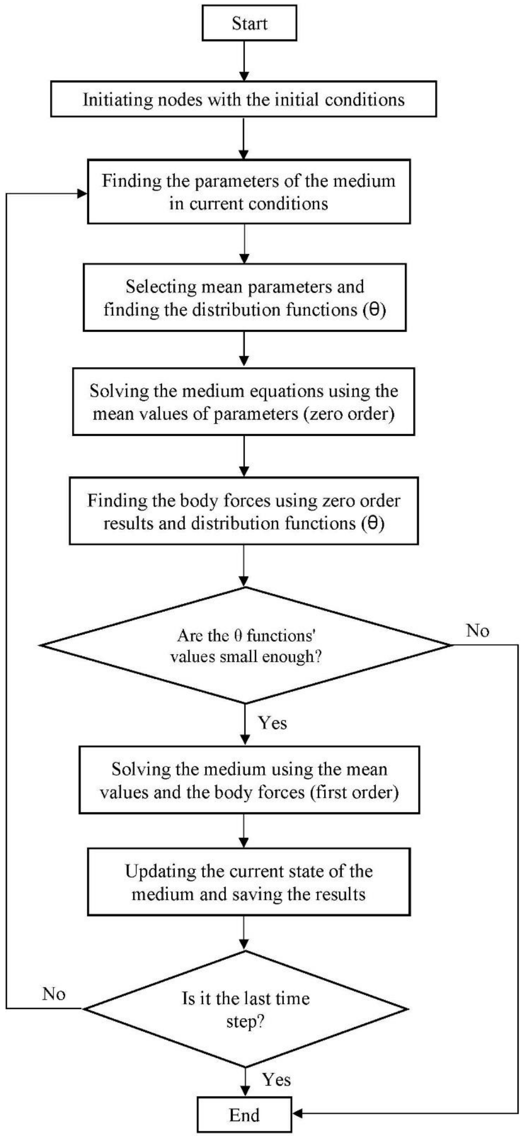

Figure 1: The Flowchart of the computational algorithm.

3. Boundary conditions

The boundary conditions are necessary for a set of differential equations to be solved. The governing differential equations system has been replaced with two new sets of differential equations so two sets of boundary conditions are needed. Here, the boundary conditions have been restricted to the Dirichlet and the Neumann types. Though, the extension to the mixed type is easy.

The zero order equations should satisfy the Dirichlet type boundary conditions of the problem. Consequently, the first order equations need to satisfy the homogeneous version of this type of boundary conditions.

For the Neumann type boundary conditions, especially when some parameters of the medium are needed (for instance, when the boundary condition has stipulated a non-zero value for the

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discharge) the local value of parameters are needed. Once more, it has been assumed that the zero order equations which emulate the response of a homogeneous medium, should meet the boundary conditions when the parameters found from such a medium. Thus, the first order equations need a new set of boundary conditions to omit the error caused by the boundary conditions of the zero order equations. It causes a new set of boundary conditions that has been developed here. The following boundary condition may be assumed:

$$dq_w = \bar{q}_w \text{ on } \Gamma_1 \tag{30}$$

which may be written as:

$$-\frac{K_w(0)}{\gamma_w}\left(1+\varepsilon\theta_{K^w}\right)\left(dP0^w+\varepsilon dP1^w\right)_{,i} n_i = d\bar{q}_w \text{ on } \Gamma_1 \tag{31}$$

$$-\frac{K_w(0)}{\gamma_w} dP0^w_{,i} n_i = dq_w(0) = d\bar{q}_w \text{ on } \Gamma_1 \tag{32}$$

$$dP1^w_{,i} n_i = -\theta_{K^w} dP0^w_{,i} \text{ on } \Gamma_1 \tag{33}$$

Where the first boundary condition (eq. (32)) has been satisfied by zero order equations and the last one (eq. (33)) should be satisfied by the first order equations. This new type of boundary conditions, together with the first order equations could be solved now.

4. Boundary Integral Equations

A set of boundary integral form of the governing equations is crucial for implementing the BEM. After finding the weak form of the equations in the weighted residual method, two distinct strategies are available to achieve a set of boundary integral equations. Some methods have been found by taking the weight functions as fundamental solutions of the governing equations [17], [18]. This procedure is named the convolution method [19]. The other strategy uses the fundamental solutions of the adjoint operator of the main problem. Such a strategy which could be seen in [20] and [6] is known as the correlation method [19]. Then the desired boundary integral equations may be found by choosing the collocation points on the boundaries.

Later, two sets of equations were derived which lead to various parts of the solution as they have been defined in (17) to (19). These equations are stipulated in (21) to (23) and (24) to (26) but, it could be seen that the equations are similar, but the boundary conditions are different. Therefore, a set of equation should be investigated, and a set of matrices will be needed for both equation sets. Although, the right-hand side of the equations need to be prepared distinctly for each set of equations.

The details of the conversion of the set of differential equations could be found in [12], where a set of dynamic type of the problem has been converted to a set of convolution type boundary integral equations. After dropping the acceleration terms of the equations and changing some constants with their counterpart in the current problem we have [12] :

$$\begin{aligned}
& s\int_\Gamma\left(d\tilde{T}_i\hat{\tilde{u}}_i - \hat{\tilde{T}}_i d\tilde{u}_i\right)d\Gamma + \int_\Gamma\left(\hat{\tilde{P}}^a d\tilde{Q}^a - d\tilde{P}^a\hat{\tilde{Q}}^a\right)d\Gamma + \int_\Gamma\left(\hat{\tilde{P}}^w d\tilde{Q}^w - d\tilde{P}^w\hat{\tilde{Q}}^w\right)d\Gamma \\
& - s\int_\Omega\left(\hat{\tilde{b}}_i d\tilde{u}_i\right)d\Omega + \int_\Omega\left(d\tilde{P}^a\hat{\tilde{\gamma}}^a - \hat{\tilde{P}}^a d\hat{\tilde{\gamma}}^a\right)d\Omega + \int_\Omega\left(d\tilde{P}^w\hat{\tilde{\gamma}}^w - \hat{\tilde{P}}^w d\hat{\tilde{\gamma}}^w\right)d\Omega \\
& = \int_\Omega\left(s\tilde{F}_i\hat{\tilde{u}}_i + F_a\hat{\tilde{P}}^a + F_w\hat{\tilde{P}}^w\right)d\Omega
\end{aligned} \tag{34}$$

where:

$$\hat{\tilde{T}}_i = \left(\lambda(0)\delta_{ij}\hat{\tilde{u}}_{k,k} + \mu(0)\left(\hat{\tilde{u}}_{i,j} + \hat{\tilde{u}}_{j,i}\right) - (1 - Sr(0))\hat{\tilde{P}}^a - Sr(0)\hat{\tilde{P}}^w \right) n_j \tag{35}$$

$$\hat{\tilde{Q}}^a = -K^a(0)\hat{\tilde{P}}^a_{,j} n_j \tag{36}$$

$$\hat{\tilde{Q}}^w = -K^w(0)\hat{\tilde{P}}^w_{,j} n_j \tag{37}$$

The right-hand side domain integral is equal to zero for zero order equations, but it should be found for first order equations where $\tilde{F}_\alpha$ could be found in equations (27) to (29).

To have a time domain reciprocal integral, an inverse Laplace transform should be implemented. So, all products change to the convolution products:

$$\begin{aligned} &\int_\Gamma \left(dT_i \times \hat{u}_i - \hat{T}_i \times d\tilde{u}_i \right) d\Gamma + \int_\Gamma \left(\hat{P}^a \times dQ^a - dP^a \times \hat{Q}^a \right) d\Gamma \\ &\qquad + \int_\Gamma \left(\hat{P}^w \times dQ^w - dP^w \times \hat{Q}^w \right) d\Gamma \\ &- \int_\Omega \left(\hat{b}_i \times du_i \right) d\Omega + \int_\Omega \left(dP^a \times \hat{\gamma}^a - P^a \times d\hat{\gamma}^a \right) d\Omega + \int_\Omega \left(dP^w \times \hat{\gamma}^w - \hat{P}^w \times d\gamma^w \right) d\Omega \\ &= \int_\Omega \left(\dot{F}_i \times \hat{u}_i + F_a \times \hat{P}^a + F_w \times \hat{P}^w \right) d\Omega \end{aligned} \tag{38}$$

To have a boundary integral equation some domain integrals should be omitted except for the right-hand side domain integral. So, it is needed to $d\gamma^w = d\gamma^a = 0$ which removes the bothering domain integrals.

Now the case $\hat{b}_i = \delta(x)H(t)$ and $\hat{\gamma}^a = \hat{\gamma}^w = 0$ will be considered in which $H(t)$ stands for the Heaviside step function and $\delta(x)$ is the Dirac Delta function. The fundamental solutions and related traction and fluxes are specified with a hat symbol. These replacements change (22) to:

$$\begin{aligned} &\int_\Gamma \left(\mathrm{d}T_i \times \hat{u}_i - \hat{T}_i \times \mathrm{d}\tilde{u}_i \right) \mathrm{d}\Gamma + \int_\Gamma \left(\hat{P}^a \times \mathrm{d}Q^a - \mathrm{d}P^a \times \hat{Q}^a \right) \mathrm{d}\Gamma \\ &\qquad + \int_\Gamma \left(\hat{P}^w \times \mathrm{d}Q^w - \mathrm{d}P^w \times \hat{Q}^w \right) \mathrm{d}\Gamma \\ &- \int_\Omega \left(\dot{F}_i \times \hat{u}_i + F_a \times \hat{P}^a + F_w \times \hat{P}^w \right) \mathrm{d}\Omega = \mathrm{d}u_i \end{aligned} \tag{39}$$

Same procedure for $\hat{\gamma}^a = \delta(x)\delta(t)$ and then $\hat{\gamma}^w = \delta(x)\delta(t)$ will results:

$$\begin{aligned} &\int_\Gamma \left(dT_i \times \hat{u}_i - \hat{T}_i \times d\tilde{u}_i \right) d\Gamma + \int_\Gamma \left(\hat{P}^a \times dQ^a - dP^a \times \hat{Q}^a \right) d\Gamma \\ &\qquad + \int_\Gamma \left(\hat{P}^w \times dQ^w - dP^w \times \hat{Q}^w \right) d\Gamma \\ &- \int_\Omega \left(\dot{F}_i \times \hat{u}_i + F_a \times \hat{P}^a + F_w \times \hat{P}^w \right) d\Omega = dP^a \end{aligned} \tag{40}$$

$$\int_{\Gamma} \left(dT_i \times \hat{u}_i - \hat{\tilde{T}}_i \times d\tilde{u}_i\right) d\Gamma + \int_{\Gamma} \left(\hat{P}^a \times dQ^a - dP^a \times \hat{Q}^a\right) d\Gamma + \int_{\Gamma} \left(\hat{P}^w \times dQ^w - dP^w \times \hat{Q}^w\right) d\Gamma - \int_{\Omega} \left(\dot{F}_i \times \hat{u}_i + F_a \times \hat{P}^a + F_w \times \hat{P}^w\right) d\Omega = dP^w \tag{41}$$

Now equations (36), (37) and (38) form a set of boundary integral equations. Having fundamental solutions in hand, a standard time domain BEM procedure could be started here for du_i, dP^a and dP^w to be found on the boundaries and then on the domain points. This procedure should be done for both zero and first order equations. Besides, after implementing the boundary conditions, the right-hand side of the first order equations is involved with the first and second order partial derivatives of the zero order solutions on the domain. It could be a weighty extra step, but it might be removed by calculation of enough data in zeroth order step and using the interpolation functions and derivations according to techniques that have been used in meshless methods [21].

The other choice to have a reciprocity type of boundary integral equations is a correlation type boundary integral equation which could be found in the dynamic version of the problem in hand [8]. After omitting the acceleration terms and converting the equal parameters name, it leads to:

$$\int_{\Gamma} \left(d\tilde{T}_i \hat{\tilde{u}}_i - \hat{\tilde{T}}_i d\tilde{u}_i \right) d\Gamma - \int_{\Gamma} \left(\hat{\tilde{P}}^a d\tilde{Q}^a - d\tilde{P}^a \hat{\tilde{Q}}^a \right) d\Gamma - \int_{\Gamma} \left(\hat{\tilde{P}}^w d\tilde{Q}^w - d\tilde{P}^w \hat{\tilde{Q}}^w \right) d\Gamma = \int_{\Omega} \left(\tilde{F}_i \hat{\tilde{u}}_i + F_a \hat{\tilde{P}}^a + F_w \hat{\tilde{P}}^w \right) d\Omega \tag{42}$$

This equation could be transformed to a boundary integral equation in which $\tilde{u}_i$, $\tilde{P}^a$ and $\tilde{P}^w$ are the fundamental solutions of the adjoint operator of which leads to the governing equations. The tractions induced by the fundamental solutions could be found as:

$$\hat{\tilde{T}}_i = \left(\lambda(0)\delta_{ij}\hat{\tilde{u}}_{k,k} + \mu(0)\left(\hat{\tilde{u}}_{i,j} + \hat{\tilde{u}}_{j,i}\right) + \left(1 - Sr(0)\right)\hat{\tilde{P}}^a + Sr(0)\hat{\tilde{P}}^w \right) n_j \tag{43}$$

$$\hat{\tilde{Q}}^a = K^a(0)\hat{\tilde{P}}^a_{,j} n_j \tag{44}$$

$$\hat{\tilde{Q}}^w = K^w(0)\hat{\tilde{P}}^w_{,j} n_j \tag{45}$$

These yield to the following boundary integral equations:

$$\int_{\Gamma} (dT_i \times \hat{u}_i - \hat{\tilde{T}}_i \times d\tilde{u}_i)\, d\Gamma - \int_{\Gamma} (\hat{P}^a \times dQ^a + dP^a \times \hat{Q}^a)\, d\Gamma - \int_{\Gamma} (\hat{P}^w \times dQ^w + dP^w \times \hat{Q}^w)\, d\Gamma - \int_{\Omega} (F_i \times \hat{u}_i + F_a \times \hat{P}^a + F_w \times \hat{P}^w)\, d\Omega = du_i \tag{46}$$

$$\int_{\Gamma} (dT_i \times \hat{u}_i - \hat{\tilde{T}}_i \times d\tilde{u}_i)\, d\Gamma - \int_{\Gamma} (\hat{P}^a \times dQ^a + dP^a \times \hat{Q}^a)\, d\Gamma - \int_{\Gamma} (\hat{P}^w \times dQ^w + dP^w \times \hat{Q}^w)\, d\Gamma - \int_{\Omega} (F_i \times \hat{u}_i + F_a \times \hat{P}^a + F_w \times \hat{P}^w)\, d\Omega = dP^a \tag{47}$$

$$\int_{\Gamma} (dT_i \times \hat{u}_i - \hat{\tilde{T}}_i \times d\tilde{u}_i)\, d\Gamma - \int_{\Gamma} (\hat{P}^a \times dQ^a + dP^a \times \hat{Q}^a)\, d\Gamma - \int_{\Gamma} (\hat{P}^w \times dQ^w + dP^w \times \hat{Q}^w)\, d\Gamma - \int_{\Omega} (F_i \times \hat{u}_i + F_a \times \hat{P}^a + F_w \times \hat{P}^w)\, d\Omega = dP^w \tag{48}$$

Obviously, the right-hand side of equation (42) is equal to zero for the zero-order solution, but like the convolution type integrals for the first order equations, they should be calculated when the zero-order solution has been found.

5. Numerical Example

In order to show the effects of the introduced body force, a 3D model has been defined whit 7×11×7 m with the data found from a real clay sample. The medium has been consolidated under its weight for a long time, so the initial conditions could be found using the properties of the soil including the special gravity and saturation ratio which varies from 50% in the surface to 64% in the depth of 7 m. A 10KN point load has been applied to the semi-infinite model and the zero order equations has been solved for a time step equal to 8640 seconds (0.1 day). Then the distribution of the characteristic parameters of the medium and finally the extra body force has been found and demonstrated. It is important to note that a small part of the medium needs extra attention for the first order solution which saves the main advantage of the boundary elements method to solve the problems in infinite and semi-infinite regions.

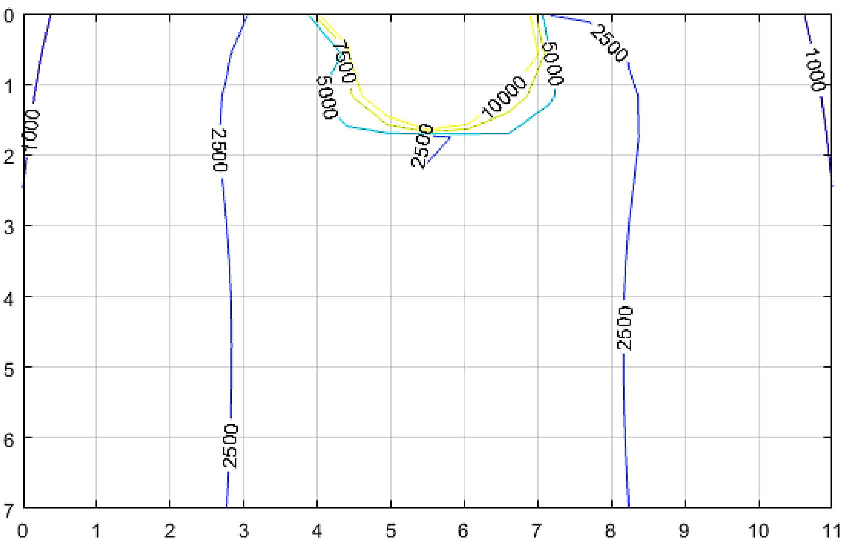

Figure 2: The body force distribution under 10 KN point load after 8640 seconds.

6. Conclusion

The unsaturated porous media experience an inhomogeneous condition during loading and deformation. Therefore, the complex and problem dependent changes in the parameters of the media make it impossible to derive the required fundamental solutions for the BIEs to be constituted. In this regard, a new formulation has been introduced for the medium that experiences a slight change in the parameters using the fundamental solutions of a homogeneous porous medium. It has been shown that the variation in the parameters of the medium could be found in terms of the variation of the state parameters. In addition, the considered variations are of the same order of variations of the state parameters. Then, a first order perturbation expansion has been employed for converting the set of equations with coordinates' dependent parameters to two sets of equations with constant parameters. This procedure requires the fundamental solutions of the problem to be derived while the parameters are constant, which consequently necessitates extra efforts, including two times of the standard BEM procedure, assessment of the second order derivatives of the solutions of the zero order equations and some domain integrals. Therefore, using the introduced BIEs and fundamental solutions for the unsaturated porous media, one can prepare the BEM numerical model, verify and compare the results with other numerical methods.

Nomenclature:

x	:	position vector
σ_{ij}	:	the stress tensor
u_i	:	displacement in direction i
λ	:	Lame's constant for the skeleton
μ	:	Lame's constant for the skeleton
P^a	:	gas (air) phase pressure
P^w	:	liquid (water) phase pressure

D_s	:	the isotropic Biot's coefficient for the fluid phase
D_{ijkl}	:	the elasticity coefficients
Sr	:	saturation ratio
α	:	constant in saturation ratio
β	:	constant in saturation ratio
K^{α}	:	coefficients of permeability for air and water
K	:	Bulk modulus for the skeleton
E	:	Young modulus for the skeleton
γ	:	specific weight
n	:	porosity of soil
n_j	:	the normal vector
$\delta(x)$	:	the Dirac delta function
$H(t)$	:	the Heaviside step function
s	:	the Laplace's transformation parameter
ε	:	symbol for small values
$\theta_{\alpha}(x)$	:	distribution function for parameter α
$\hat{\tilde{T}}_i$	:	traction induced by the fundamental solutions in the Laplace's transformation space
$\hat{\tilde{Q}}^{\alpha}$	:	flux of phase α, induced by the fundamental solutions in the Laplace's transformation space
c_{ij}	:	intermediate coefficients
q_i^{α}	:	the fluid volume fluxes for air and water

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Appendix A

It has been claimed that any variation in the parameters of the medium could be found in terms of variations of the state parameters. In this appendix it has been explained for different parameters. A dimensionless type of the state parameters could be defined:

$$\frac{P^a(x)}{P^c(0)}=\frac{P^a(0)}{P^c(0)}+\varepsilon\theta_a(x) \tag{A-1}$$

$$\frac{P^w(x)}{P^c(0)}=\frac{P^w(0)}{P^c(0)}+\varepsilon\theta_w(x) \tag{A-2}$$

$$\frac{\sigma'_{ij}(x)}{P^c(0)}=\frac{\sigma_{ij}(x)-P^a(x)}{P^c(0)}=\frac{\sigma'_{ij}(x)}{P^c(0)}+\varepsilon\theta_\sigma(x) \tag{A-3}$$

$$\begin{aligned}\frac{P^c(x)}{P^c(0)}&=\frac{P^a(x)-P^w(x)}{P^c(0)}=\frac{P^a(0)}{P^c(0)}+\varepsilon\theta_a(x)-\frac{P^w(0)}{P^c(0)}-\varepsilon\theta_w(x)\\&=1+\varepsilon\left(\theta_a(x)-\theta_w(x)\right)=1+\varepsilon\theta_c(x)\end{aligned} \tag{A-4}$$

The state surface for the void ratio is [22]:

$$\frac{1+e(x)}{1+e(0)}=\exp\left(B(x)\right) \qquad \text{(A-5)}$$

Where:

$$B(x)=\frac{1}{(1-m)K_b P_{atm}^{1-m}}\left(a\sigma'(x)+bP^c(x)\left(1-\frac{\sigma'(x)}{\sigma_e}\right)\right)^{1-m} \qquad \text{(A-6)}$$

In which a , b , m , K_b and P_{atm} are materials constants. Using (A-1) to (A-3) in (A-6):

$$B(x)=\frac{P^c(0)}{(1-m)K_b P_{atm}^{1-m}}\times$$

$$\left(a\frac{\sigma'(0)}{P^c(0)}+b\left(1-\frac{\sigma'(0)}{\sigma_e}\right)+\varepsilon\left(a\theta_\sigma(x)+b\left(\theta_c(x)\left(1-\frac{\sigma'(0)}{\sigma_e}\right)\right)-\frac{P^c(0)}{\sigma_e}\theta_\sigma(x)\right)\right)^{1-m} \qquad \text{(A-7)}$$

Finally, using Taylor's expansion and dropping the infinitesimal terms, then replacing the result in (A-5):

$$1+e(x)=\left(1+e(0)\right)\left(1+\varepsilon\theta_B(x)\right) \qquad \text{(A-8)}$$

$$\theta_B(x)=(m-1)\left(a\theta_\sigma(x)+b\left(\theta_c(x)\left(1-\frac{\sigma'(0)}{\sigma_e}\right)\right)-\frac{P^c(0)}{\sigma_e}\theta_\sigma(x)\right)\left(a\frac{\sigma'(0)}{P^c(0)}+b\left(1-\frac{\sigma'(0)}{\sigma_e}\right)\right)^{-m} \qquad \text{(A-9)}$$

The relationship between e and n (porosity) results in a similar equation for n:

$$n(x)=n(0)\left(1+\varepsilon\theta_n(x)\right) \qquad \text{(A-10)}$$

$$\theta_n(x)=\theta_B(x)\frac{1-n(0)}{n(0)} \qquad \text{(A-11)}$$

The water retention curve could be used to derive a similar equation for the saturation ratio:

$$Sr(x)=Sr(0)\left(1+\varepsilon\theta_{Sr}(x)\right) \qquad \text{(A-12)}$$

$$\theta_{Sr}(x)=\frac{\beta\theta_c(x)}{Sr(0)} \qquad \text{(A-13)}$$

Where β has been used to express the saturation ratio in terms of suction.

The air and water permeability could be explained in the same form:

$$K_a(x)=K_a(0)\left(1+\varepsilon\theta_{K_a}(x)\right) \qquad \text{(A-14)}$$

$$\theta_{K_a}(x)=E_k\left(\theta_B(x)\frac{1+e(0)}{e(0)}-\theta_{Sr}(x)\frac{Sr(0)}{1-Sr(0)}\right) \qquad \text{(A-15)}$$

$$K_w(x) = K_w(0)\left(1 + \varepsilon\theta_{K_w}(x)\right) \tag{A-16}$$

$$\theta_{K_w}(x) = 3{,}5\theta_B(x)\frac{Sr(0)}{1 - Sr_u} \tag{A-17}$$

The constants have been defined later in the paper.

The tangential elastic module could be evaluated using the Kondner's hyperbolic law for unsaturated soils [22]:

$$E_t(x) = \left(E_e(x) + E_s(x)\right)\left(1 - R_f S(x)\right) \tag{A-18}$$

$$S(x) = \frac{\sigma_1 - \sigma_3}{\left(\sigma_1 - \sigma_3\right)_{ult}} = S(0)\ \left(1 + \varepsilon\theta_S(x)\right) \tag{A-19}$$

$$E_e(x) = KP_{atm}\left(\frac{\sigma_3}{P_{atm}}\right)^n \tag{A-20}$$

$$E_s(x) = m_1 P^c \tag{A-21}$$

$$E_t(x) = E_t(0)\ \left(1 + \varepsilon\theta_{E_t}(x)\right) \tag{A-22}$$

$$\theta_{E_t}(x) = \frac{nE_e(0)}{E_e(0) + E_s(0)}\theta_{\sigma_3}(x) + \frac{E_s(0)}{E_e(0) + E_s(0)}\theta_c(x) - \frac{R_f S(0)}{1 - R_f S(0)}\theta_S(x) \tag{A-23}$$

The bulk module could be found using the Kondner's hyperbolic law and the state surface which has been stipulated previously [22]:

$$K(x) = \frac{K_b}{P_{atm}^{1-m}\left(a - \frac{b}{\sigma_e}P^c(x)\right)}B(x)^m \tag{A-24}$$

Using the spatial assessment for B and P^c and omitting the terms that includes second and higher orders of ε leads to:

$$K(x) = K(0)\left(1 + \varepsilon\theta_K(x)\right) \tag{A-25}$$

$$\theta_K(x) = \frac{m\left(a\theta_\sigma(x) + b\left(\theta_c(x)\left(1 - \frac{\sigma'(0)}{\sigma_e}\right)\right) - \frac{P^c(0)}{\sigma_e}\theta_\sigma(x)\right)}{\left(a\frac{\sigma'(0)}{P^c(0)} + b\left(1 - \frac{\sigma'(0)}{\sigma_e}\right)\right)} + \frac{bP^c(0)}{\sigma_e\left(a - \frac{bP^c(0)}{\sigma_e}\right)}\theta_c(x) \tag{A-26}$$

Finally using the known relations to convert the bulk and the elastic moduli to the Lame's coefficients leads to:

$$\lambda(x) = \lambda(0)\ \left(1 + \varepsilon\theta_\lambda(x)\right) \tag{A-27}$$

$$\theta_{\lambda}(x)=\frac{3K(0)\ \left(-6K(0)E_t(0)\theta_{E_t}(x)+\left(E_t(0)^2-6K(0)E_t(0)+27K(0)^2\right)\ \theta_K(x)\right)}{\left(9K(0)-E_t(0)\right)^2} \tag{A-28}$$

$$\mu(x)=\mu(0)\ \left(1+\varepsilon\theta_{\mu}(x)\right) \tag{A-29}$$

$$\theta_{\mu}(x)=\frac{3K(0)\ E_t(0)\left(9K(0)\ \theta_{E_t}(x)-E_t(0)\ \theta_K(x)\right)}{\left(9K(0)-E_t(0)\right)^2} \tag{A-30}$$

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